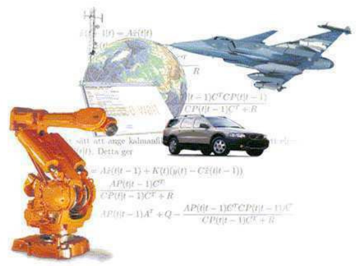
Linear Systems

Lecture 7. Smith form, state space, zeros and poles



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Matrix Fraction Descriptions (MFDs)

$$G(s) = N_R(s)D_R^{-1}(s) \text{ or } G(s) = D_L^{-1}(s)N_L(s)$$

where N_R, D_R, N_L, D_L are *polynomial matrices*.

- $U(s)$ unimodular: U^{-1} also polynomial $\Leftrightarrow \det U(s)$ constant $\neq 0$
- right (left) coprime matrices: no common right (left) factors (except unimodular ones)
- $N_1(s)D_1^{-1}(s) = N_2(s)D_2^{-1}(s)$; N_1, D_1 and N_2, D_2 coprime $\Rightarrow$ exists unimodular U so that $N_1(s) = N_2(s)U(s), \quad D_1(s) = D_2(s)U(s)$

A fundamental invariant

Let $G(s)$ be given

For **coprime** realizations

$$G(s) = N_R(s)D_R(s)^{-1} \quad \text{or} \quad G(s) = D_L(s)^{-1}N_L(s)$$

the polynomial

$$p(s) = \det D_R(s) \quad \text{or} \quad p(s) = \det D_L(s)$$

is always the same (except for multiplication with a constant)

Corresponding result for N_L, N_R more difficult (not necessarily square matrix)

McMillan degree

For a strictly proper $G(s)$ with **coprime** realizations

$$G(s) = N_R(s)D_R(s)^{-1} \quad \text{or} \quad G(s) = D_L(s)^{-1}N_L(s)$$

the number

$$m_c = \text{deg} \det D_R = \text{deg} \det D_L$$

is called the **McMillan degree** of G.
Theorem. If G has a realization with dimension n then

$$n \geq m_c$$

Realization idea

Generalize the controller form for scalar transfer functions.

$$G(s) = \frac{c_{n-1}s^{n-1} + \cdots + c_1 s + c_0}{s^n + a_{n-1} + \cdots + a_1 s + a_0}$$

has the realization

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 1 \\ -a_0 & -a_1 & & \dots & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_0 & c_1 & & \dots & c_{n-1} \end{bmatrix} x$$

Preparing the transfer function

$G(s) = N(s)D(s)^{-1}$, D column reduced with column degrees $k_1,..,k_m$. Then D and N can be written

$$D(s) = D_{hc}\Delta(s) + D_\ell \psi(s), \quad N(s) = N_\ell \psi(s)$$

for some constant matrices D_ℓ and N_ℓ

$$\Delta(s) = \begin{bmatrix} s^{k_1} & 0 & \dots & 0 \\ 0 & s^{k_2} & \dots & 0 \\ \vdots & & \ddots & \\ 0 & & & s^{k_m} \end{bmatrix}, \quad \psi(s) = \begin{bmatrix} \psi_{k_1} & 0 & \dots & 0 \\ 0 & \psi_{k_2} & \dots & 0 \\ \vdots & & \ddots & \\ 0 & & & \psi_{k_m} \end{bmatrix}$$

$$\psi_k(s) = \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{k-1} \end{bmatrix}$$

Skeleton state space matrices

$$A_0 = \begin{bmatrix} A_{k_1} & 0 & \dots & 0 \\ 0 & A_{k_2} & \dots & 0 \\ \vdots & & \ddots & \\ 0 & & & A_{k_m} \end{bmatrix}, \quad B_0 = \begin{bmatrix} B_{k_1} & 0 & \dots & 0 \\ 0 & B_{k_2} & \dots & 0 \\ \vdots & & \ddots & \\ 0 & & & B_{k_m} \end{bmatrix}$$

where

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 1 \\ 0 & 0 & & \dots & 0 \end{bmatrix}_{k \times k}, \quad B_k = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{k \times 1}$$

A minimal realisation for right matrix fractions

Theorem $G(s) = N(s)D(s)^{-1} = (N_\ell \psi(s))(D_{hc}\Delta(s) + D_\ell \psi(s))^{-1}$ has a realisation

$$\dot{x} = (A_0 - B_0 D_{hc}^{-1} D_\ell)x + B_0 D_{hc}^{-1} u, \quad y = N_\ell x$$

The dimension of the state space is $n = k_1 + \cdots + k_m$.

Corollary 1 If N and D are coprime the realization is minimal with n = McMillan degreee of G.

Corollary 2 For an arbitrary strictly proper transfer function the dimension of a minimal realization equals the McMillan degree.

Realization. An example

In a previous example we had the right denominator matrix $D_R =$

$$\begin{bmatrix} s^2+3s+2 & -s-2 \\ 0 & s+2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}_{D_{hc}} \underbrace{\begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix}}_{\Delta(s)} + \underbrace{\begin{bmatrix} 2 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix}}_{D_\ell} \underbrace{\begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \end{bmatrix}}_{\psi(s)}$$

This gives

$$A = A_0 - B_0 D_{hc}^{-1} D_\ell = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad B = B_0 D_{hc}^{-1} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Realization example, cont'd.

The numerator matrix is

$$N_R(s) = \begin{bmatrix} 2+s & -1 \\ s & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}}_{N_\ell} \underbrace{\begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \end{bmatrix}}_{\psi(s)}$$

giving

$$C = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

Poles and zeros

Wanted: A tool for determining system properties (poles, zeros, controllability, observability,....) for a system in any representation (state space, transfer function, matrix fraction, DAE,...)

Answer: Smith form

Smith form

Kailath: pp 390-392
Smith form of rank r polynomial matrix $P(s)$

$$U(s)P(s)V(s) = \begin{pmatrix} \lambda_1(s) & 0 & 0 & \dots & & 0 \\ 0 & \lambda_2(s) & 0 & \dots & & 0 \\ 0 & 0 & \ddots & & & \vdots \\ \vdots & \vdots & & \lambda_r(s) & & \\ & & & & 0 & \\ & & & & & \ddots \\ 0 & 0 & \dots & & & 0 \end{pmatrix}$$

U, V unimodular

$\lambda_i(s)$ divides $\lambda_{i+1}(s)$, invariant polynomials

Invariant polynomials and minors

$P(s)$ polynomial matrix, rank = r

$$\Delta_i = \text{gcd of all } i \times i \text{ minors of } P(s)$$

If $Q(s) = U(s)P(s)V(s)$, U, V unimodular, then Q and P have the same Δ_i.

$$\Delta_1(s) \text{ divides } \Delta_2(s) \text{ divides } \cdots \text{ divides } \Delta_r(s)$$

$$\lambda_1(s) = \Delta_1(s)$$
$$\lambda_i(s) = \frac{\Delta_i(s)}{\Delta_{i-1}(s)}, \quad i = 2, \ldots, r$$

Examples

$\rightarrow$ denotes transformation to Smith form.

$$U(s) \rightarrow I, \quad \text{if } U(s) \text{ unimodular}$$

$$\begin{pmatrix} s+2 & -1 \\ s & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & s+1 \end{pmatrix} \quad \begin{pmatrix} s & s^2 \\ s+s^3 & s^2 \end{pmatrix} \rightarrow \begin{pmatrix} s & 0 \\ 0 & s^4 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{pmatrix}, \quad sI - A \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s^3 + a_1 s^2 + a_2 s + a_3 \end{pmatrix}$$

Smith transformation of $G(s)$ (Kailath pp 443-448)

$$G(s) = \frac{1}{d(s)} N(s)$$

d: least common multiple of denominators, N polynomial matrix.
Smith form: $N(s) = U(s)\Lambda(s)V(s)$, $\quad U$, V unimodular

$$G(s) = U(s) \underbrace{\begin{pmatrix} \text{diag}\left(\frac{\epsilon_i(s)}{\psi_i(s)}\right) & 0 \\ 0 & 0 \end{pmatrix}}_{\text{Smith-McMillan form}} V(s)$$

where ϵ_i, ψ_i without common factors.

$$\frac{\epsilon_i(s)}{\psi_i(s)} = \frac{\lambda_i(s)}{d(s)}, \quad i = 1, \ldots, r, \quad \psi_{i+1}(s) | \psi_i(s), \quad \epsilon_i(s) | \epsilon_{i+1}(s),$$
$$\psi_1(s) = d(s)$$

Smith form and MFDs

It is possible to write

$$G(s) = U(s) \underbrace{\begin{pmatrix} \text{diag}(\psi_i(s)) & 0 \\ 0 & I_{p-r} \end{pmatrix}^{-1}}_{\psi_L(s)} \underbrace{\begin{pmatrix} \text{diag}(\epsilon_i(s)) & 0 \\ 0 & 0 \end{pmatrix}}_{\mathcal{E}(s)} V(s)$$

Left coprime MFD: $D_L(s) = \psi_L(s) U^{-1}(s)$, $N_L(s) = \mathcal{E}(s)V(s)$
Alternatively

$$G(s) = U(s) \underbrace{\begin{pmatrix} \text{diag}(\epsilon_i(s)) & 0 \\ 0 & 0 \end{pmatrix}}_{\mathcal{E}(s)} \underbrace{\begin{pmatrix} \text{diag}(\psi_i(s)) & 0 \\ 0 & I_{m-r} \end{pmatrix}^{-1}}_{\psi_R(s)} V(s)$$

Right coprime MFD: $D_R(s) = V^{-1}(s)\psi_R(s)$, $N_R(s) = U(s)\mathcal{E}(s)$

Invariants in transfer functions.

Let $G(s)$ have different left or right coprime MFDs. Then

- All numerator matrices have the same Smith form ($= \mathcal{E}(s)$).
- All denominator matrices have the same Smith form (except for extra 1s on the diagonal).
- The invariant polynomials of the numerators are the $\epsilon_i(s)$ of the Smith-McMillan form.
- The invariant polynomials of the denominators are the $\psi_i(s)$ of the Smith-McMillan form.

Poles and zeros

- Definition using Smith-McMillan form.
  - The **poles** of a system are the roots of the ψ_i in the Smith-McMillan form.
  - The **McMillan degree** is the degree of $\det D(s)$.
  - The **zeros** of a system are the roots of the ϵ_i in the Smith-McMillan form
- Equivalent definition
  - The **poles** of a system are the roots of $\det D(s) = 0$ where D is the denominator of **any coprime MFD**.
  - The **zeros** of a system are the s-values for which the rank of $N(s)$ drops, where N is the numerator of **any coprime MFD**.

Example

$$G(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ \frac{s}{(s+1)(s+2)} & \frac{2s+1}{(s+1)(s+2)} \end{pmatrix} = \frac{1}{(s+1)(s+2)} \begin{pmatrix} s+2 & 1 \\ s & 2s+1 \end{pmatrix}$$

The Smith McMillan form is

$$\begin{pmatrix} \frac{1}{(s+1)(s+2)} & 0 \\ 0 & \frac{s+1}{s+2} \end{pmatrix}$$

with

$$\epsilon_1 = 1,\ \epsilon_2 = s+1; \quad \psi_1 = (s+1)(s+2),\ \psi_2 = s+2$$

poles: $-1, -2, -2$
zeros: -1

Connection to state space

For a right fraction $N_R(s)D_R(s)^{-1}$ we had found a realization A_c, B_c, C_c with

$$B_c D_R(s) = (sI - A_c)\psi(s), \quad N_R(s) = C_c \psi(s)$$

There exist $X(s)$, $Y(s)$, $\tilde{X}(s)$, $\tilde{Y}(s)$ so that

$$\begin{pmatrix} sI - A_c & B_c \\ \tilde{X}(s) & \tilde{Y}(s) \end{pmatrix} \underbrace{\begin{pmatrix} X(s) & -\Psi(s) \\ Y(s) & D_R(s) \end{pmatrix}}_{\text{unimodular}} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

Smith equivalence of state space and MFDs

It follows that

$$\begin{pmatrix} sI - A_c & 0 \\ 0 & I \end{pmatrix} \overset{S}{\sim} \begin{pmatrix} I & 0 \\ 0 & D_R(s) \end{pmatrix}, \quad \begin{pmatrix} sI - A_c & B_c \\ -C_c & 0 \end{pmatrix} \overset{S}{\sim} \begin{pmatrix} I & 0 \\ 0 & N_R(s) \end{pmatrix}$$

where $\overset{S}{\sim}$ means “same Smith form”. This generalizes to

$$\begin{pmatrix} sI - A & 0 \\ 0 & I \end{pmatrix} \overset{S}{\sim} \begin{pmatrix} I & 0 \\ 0 & D(s) \end{pmatrix}, \quad \begin{pmatrix} sI - A & B \\ -C & 0 \end{pmatrix} \overset{S}{\sim} \begin{pmatrix} I & 0 \\ 0 & N(s) \end{pmatrix}$$

where A, B, C is any minimal realization of a strictly proper $G(s)$ with (left or right) coprime MFD given by $N(s)$ (numerator) and $D(s)$ (denominator).

State space characterization of poles and zeros

(minimal realization)

- the poles are given by $\det(sI - A) = 0$
- the zeros are the s-values for which the rank of

$$\begin{pmatrix} sI - A & B \\ -C & 0 \end{pmatrix}$$

  drops.
- Time domain interpretation of zeros:
  - If λ is a zero, then there exist constants x_0, $u_0 \neq 0$ so that the input $u(t) = e^{\lambda t} u_0$, together with the initial condition $x(0) = x_0$ gives the output $y(t) = 0$, $t \geq 0$.

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