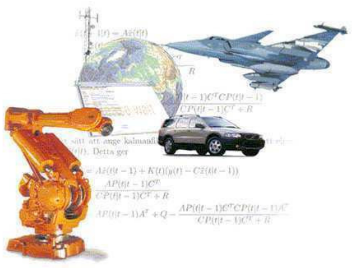
Linear Systems

Lecture 8. Linear DAE (Differential Algebraic Equations) Systems



Torkel Glad

Reglerteknik, ISY, Linköpings Universitet

Linear systems so far

Different descriptions of the same system

- Transfer function $G(s)$
  - Left MFD. $G(s) = D_L^{-1} N_L$, N_L, D_L **left coprime**
  - Right MFD. $G(s) = N_R D_R^{-1}$, N_R, D_R **right coprime**
  - State space. $G(s) = C(sI - A)^{-1}B$, A, B, C **minimal realization**

Important invariants

- $D_L, D_R, sI - A$ same Smith form: **poles**
- $N_L, N_R, \begin{bmatrix} sI - A & B \\ -C & 0 \end{bmatrix}$ same Smith form: **zeros**

General modeling

"A model is any collection of equations in differentiated and undifferentiated variables" (Modelica and similar modeling tools)

Can you define and compute the system properties for such a model in the linear case?

A general description of linear systems

Consider a physical system described by an input vector u, an output vector y and a vector of internal physical variables ζ. We assume

- u is determined externally.
- u is sufficient to define a solution for ζ (except for initial conditions)
- ζ in itself is unimportant; we can add or delete variables and transform them.
- If it is important to keep track of a certain physical variable, it is included in y.
- u and y are not transformed.

The PMD description

Assuming that all relations between the variables and their derivatives are linear we arrive at a representation of the form

$$P(s)\zeta = Q(s)u$$
$$y = R(s)\zeta + W(s)u$$

where P, Q, R and W are polynomial matrices.
Interpretation of s:

- $\frac{d}{dt}$ (continuous time)
- complex number (continuous time, Laplace transform)
- shift operator: $\zeta(t) \to \zeta(t+1)$ (discrete time)
- complex number (discrete time, z-transform)

PMD description, cont'd.

Matrix notation:

$$\underbrace{\begin{bmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{bmatrix}}_{\mathbf{P}} \begin{bmatrix} -\zeta \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$$

$\mathbf{P}$ is called the **system matrix**.
$P(s)$ is usually assumed to be invertible (ζ uniquely determined by u)
The transfer function is

$$G(s) = R(s)P(s)^{-1}Q(s) + W(s)$$

Special cases

Right fraction $y = N_R D_R^{-1} u$: $\mathbf{P} = \begin{bmatrix} D_R(s) & I \\ -N_R(s) & 0 \end{bmatrix}$

Left fraction $y = D_L^{-1} N_L u$: $\mathbf{P} = \begin{bmatrix} D_L(s) & N_L(s) \\ I & 0 \end{bmatrix}$

State space: $\mathbf{P} = \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix}$

DAE (Descriptor): $\mathbf{P} = \begin{bmatrix} sE - A & B \\ -C & D \end{bmatrix}$

Transformation of ζ-equations

- change the order
- multiply one equation with nonzero constant
- add one equation multiplied by a polynomial to another equation.

If a pair ζ, u is a solution before one of these transformations is made, it is still a solution afterwards and vice versa.
These row operations correspond to a multiplication from the left:

$$\begin{bmatrix} M(s) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{bmatrix}$$

where M is unimodular.

Transformation of y-equations

Since we do not transform y, the only possible change to a y-equation is to add a polynomial multiple of a ζ-equation.

In matrix terms such transformations are described by

$$\begin{bmatrix} I & 0 \\ X(s) & I \end{bmatrix} \begin{bmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{bmatrix}$$

where X is a polynomial matrix.

Transformation of ζ

- Multiply a variable with a nonzero constant.
- Let two variables change places.
- Add a polynomial multiple of a variable to another one.

These transformations correspond to multiplication by a unimodular matrix $\bar{M}(s)$:

$$\bar{\zeta} = \bar{M}(s)\zeta$$

If one allows addition of polynomial multiples of u the transformation becomes

$$\begin{bmatrix} -\bar{\zeta} \\ u \end{bmatrix} = \begin{bmatrix} \bar{M}(s) & \bar{Y}(s) \\ 0 & I \end{bmatrix} \begin{bmatrix} -\zeta \\ u \end{bmatrix}$$

Transformation of ζ, cont'd.

The inverse transformation is

$$\begin{bmatrix} -\zeta \\ u \end{bmatrix} = \begin{bmatrix} M(s) & Y(s) \\ 0 & I \end{bmatrix} \begin{bmatrix} -\bar{\zeta} \\ u \end{bmatrix}$$

where $M = \bar{M}^{-1}$, $Y = -\bar{M}^{-1}\bar{Y}$.
The transformation of the system matrix is thus:

$$\begin{bmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{bmatrix} \begin{bmatrix} M(s) & Y(s) \\ 0 & I \end{bmatrix}$$

with M unimodular and Y polynomial

Equivalence

The previous reasoning makes the following definition natural:
Two systems are **equivalent** if there are unimodular matrices M_1, M_2 and polynomial matrices X, Y such that the system matrices are related as

$$\underbrace{\begin{bmatrix} M_1(s) & 0 \\ X(s) & I \end{bmatrix}}_{U_1} \underbrace{\begin{bmatrix} P_1(s) & Q_1(s) \\ -R_1(s) & W_1(s) \end{bmatrix}}_{\mathbf{P}_1} \underbrace{\begin{bmatrix} M_2(s) & Y(s) \\ 0 & I \end{bmatrix}}_{U_2} = \underbrace{\begin{bmatrix} P_2(s) & Q_2(s) \\ -R_2(s) & W_2(s) \end{bmatrix}}_{\mathbf{P}_2}$$

Since U_1, U_2 are unimodular we have

$$\mathbf{P}_1 \overset{S}{\sim} \mathbf{P}_2, \quad P_1 \overset{S}{\sim} P_2, \quad \begin{bmatrix} P_1 & Q_1 \end{bmatrix} \overset{S}{\sim} \begin{bmatrix} P_2 & Q_2 \end{bmatrix}, \quad \begin{bmatrix} P_1 \\ -R_1 \end{bmatrix} \overset{S}{\sim} \begin{bmatrix} P_2 \\ -R_2 \end{bmatrix}$$

Equivalence and transfer function

A straightforward calculation shows that the equivalence transformation does not change the transfer function.

Example. DC-motor.

$y_1 = \zeta_1$ = motor angle,
$y_2 = \zeta_2$ = angular velocity,
u = input voltage

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= -\zeta_2 + u \\ y_1 &= \zeta_1 \\ y_2 &= \zeta_2 \end{aligned} \qquad \mathbf{P} = \begin{bmatrix} s & -1 & 0 \\ 0 & s+1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

DC motor, transformations

$$\mathbf{P} = \begin{bmatrix} s & -1 & 0 \\ 0 & s+1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & s & 0 \\ -(s+1) & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & s & 0 \\ 0 & s(s+1) & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \to$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & s(s+1) & 1 \\ 0 & -1 & 0 \\ 1 & -s & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & s(s+1) & 1 \\ 0 & -1 & 0 \\ 0 & -s & 0 \end{bmatrix} \to \begin{bmatrix} s(s+1) & 1 \\ -1 & 0 \\ -s & 0 \end{bmatrix}$$

The result is a matrix fraction description:

$$G = \begin{bmatrix} 1 & s \end{bmatrix} (s^2+s)^{-1}$$

Rosenbrock equivalence

To be really useful the equivalence concept has to be extended so that the following system matrices are regarded as equivalent

$$\begin{bmatrix} I & 0 & 0 \\ 0 & P(s) & Q(s) \\ 0 & -R(s) & W(s) \end{bmatrix}, \quad \begin{bmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{bmatrix}$$

where the unit matrix is of arbitrary dimension.

- This corresponds to addition or deletion of trivial equations of the form $\zeta_i = 0$, where ζ_i does not occur in any other equation.
- The Smith form is only changed by the addition or deletion of trivial ones on the diagonal.

State space form

An arbitrary system matrix is equivalent to one in state space form:

$$\begin{bmatrix} sI - A & B \\ -C & J(s) \end{bmatrix}$$

This can be seen by using two facts:
(I) For an arbitrary matrix $\Lambda(s)$ in Smith form it is possible to find a constant matrix A and unimodular matrices $U(s)$ and $V(s)$ such that

$$\Lambda(s) = U(s)(sI - A)V(s)$$

(possibly after adding or deleting ones on the diagonal of Λ)
Idea of proof: take block-diagonal A, each block a companion matrix corresponding to an invariant polynomial.

state space, cont'd.

(II) For any $P(s)$ and any A (of compatible dimensions)

$$P(s) = Q_1(s)(sI - A) + R_1$$
$$P(s) = (sI - A)Q_2(s) + R_2$$

with *constant* R_1, R_2.
Idea of proof: compare powers of s on both sides.

Transformation to state space form

1) Using (I), choose unimodular M_1 and M_2 so that

$$\begin{bmatrix} M_1(s) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{bmatrix} \begin{bmatrix} M_2(s) & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} sI - A & \tilde{Q}(s) \\ -\tilde{R}(s) & W(s) \end{bmatrix}$$

where $\tilde{R} = RM_2$, $\tilde{Q} = M_1 Q$.
2) Using (II), write

$$\tilde{R}(s) = X(s)(sI - A) + C, \quad C \text{ const.}$$

Use the transformation

$$\begin{bmatrix} I & 0 \\ X(s) & I \end{bmatrix} \begin{bmatrix} sI - A & \tilde{Q}(s) \\ -\tilde{R}(s) & W(s) \end{bmatrix} = \begin{bmatrix} sI - A & \tilde{Q}(s) \\ -C & W(s) + X(s)\tilde{Q}(s) \end{bmatrix}$$

Transformation to state space form cont'd.

3) Using (II), write

$$\tilde{Q}(s) = (sI - A)Y(s) + B, \quad B \text{ const.}$$

Use the transformation

$$\begin{bmatrix} sI - A & \tilde{Q}(s) \\ -C & W(s) + X(s)\tilde{Q}(s) \end{bmatrix} \begin{bmatrix} I & -Y(s) \\ 0 & I \end{bmatrix} = \begin{bmatrix} sI - A & B \\ -C & J(s) \end{bmatrix}$$

where $J(s) = W(s) + X(s)\tilde{Q}(s) + CY(s)$. The state space description is

$$\dot{x} = Ax + Bu, \quad y = Cx + J(d/dt)u$$

J depends on $s \Rightarrow u$ is differentiated.

Controllability and observability

Since any system can be transformed into state space form:

$$\underbrace{\begin{bmatrix} M_1(s) & 0 \\ X(s) & I \end{bmatrix}}_{U_1} \underbrace{\begin{bmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{bmatrix}}_{\mathbf{P}_1} \underbrace{\begin{bmatrix} M_2(s) & Y(s) \\ 0 & I \end{bmatrix}}_{U_2} = \underbrace{\begin{bmatrix} sI - A & B \\ -C & J(s) \end{bmatrix}}_{\mathbf{P}_2}$$

we have

$$P(s) \overset{S}{\sim} sI - A$$
$$\begin{bmatrix} P(s) & Q(s) \end{bmatrix} \overset{S}{\sim} \begin{bmatrix} sI - A & B \end{bmatrix}$$
$$\begin{bmatrix} P(s) \\ -R(s) \end{bmatrix} \overset{S}{\sim} \begin{bmatrix} sI - A \\ -C \end{bmatrix}$$

Controllability $\Leftrightarrow$ P, Q left coprime
Observability $\Leftrightarrow$ P, R right coprime

Irreducibility

A system

$$\mathbf{P} = \begin{bmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{bmatrix}$$

is called **irreducible** if P, Q are left coprime and P, R right coprime.

All state space descriptions equivalent to $\mathbf{P}$ are then controllable and observable and hence minimal.

Consequence:
All irreducible systems having the same transfer function are equivalent.

Poles and zeros

A transfer function in Smith-McMillan form:

$$G(s) = U(s) \underbrace{\begin{pmatrix} \text{diag}(\epsilon_i(s)) & 0 \\ 0 & 0 \end{pmatrix}}_{\mathcal{E}(s)} \underbrace{\begin{pmatrix} \text{diag}(\psi_i(s)) & 0 \\ 0 & I_{m-r} \end{pmatrix}}_{\psi_R(s)}^{-1} V(s)$$

system matrix: $\mathbf{P}_{McM} = \begin{bmatrix} \psi_R(s) & V(s) \\ -U(s)\mathcal{E}(s) & 0 \end{bmatrix} \overset{S}{\sim} \begin{bmatrix} I & 0 \\ 0 & \mathcal{E}(s) \end{bmatrix}$

Any other irreducible system $\mathbf{P} = \begin{bmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{bmatrix}$

having the same transfer function G must be equivalent. It follows that:
The **poles** of G are given by $\det P(s) = 0$.
The **zeros** of G are given by the invariant polynomials of $\mathbf{P}(s)$.

Input decoupling zeros

Suppose $\mathbf{P} = \begin{bmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{bmatrix}$ P, Q not coprime

Exists equivalent state space description that is uncontrollable.

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

where A_{11}, B_1 is controllable. Then

$$\begin{bmatrix} P & Q \end{bmatrix} \overset{S}{\sim} \begin{bmatrix} sI - A_{11} & -A_{12} & B_1 \\ 0 & sI - A_{22} & 0 \end{bmatrix} \overset{S}{\sim} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & sI - A_{22} \end{bmatrix}$$

The zeros of the Smith form polynomials of $\begin{bmatrix} P & Q \end{bmatrix}$ are thus the eigenvalues of A_{22}, i.e. the “uncontrollable poles”. They are called **input decoupling zeros**.

Output decoupling zeros

Similarly the Smith zeros of

$$\begin{bmatrix} P \\ -R \end{bmatrix}$$

are called **output decoupling zeros**