Homework 1

Networked Control and Multi-Agent Systems

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1.

Recall that if x_i is scalar, with its derivative given by the consensus equation

$$\dot{x}_i = \sum_{j \in N_i} (x_j - x_i),\ i = 1, \ldots, j,$$

this can be written as

$$\dot{x} = -Lx,$$

where L is the Laplacian of the network graph (assume undirected) and $x = (x_1, \ldots, x_N)^T$.

a

If instead

$$\dot{x} = -L^2 x,$$

what are the corresponding node-level dynamics, i.e., find

$$\dot{x}_i = ???.$$

b

Can you give a graph theoretic interpretation to your answer to **1a**?

2.

An undirected graph G is connected if and only if its Laplacians second smallest eigenvalue λ_2 is nonzero. Using a similar argument as the one in class, show that the number of connected components (i.e. connected subgraphs that are disconnected from each other) is equal to the number of zero eigenvalues of the Laplacian.

3.

If $G = (V, E)$ is connected, static, and undirected, then the consensus equation

$$\dot{x}_i = -\sum_{j \in N_i} (x_i - x_j),\ i = 1, \ldots, N$$

drives all agents to span$\{\mathbf{1}\}$. In particular, all agents end up in the static centroid.

One way of showing this is to let $x_i \to x^\star$, and assume that $x^\star$ is invariant and given by a linear combination of the initial conditions, i.e.

$$x^\star = \sum_{i=1}^{N} \xi_i x_i(0),$$

for some $\xi_1, \ldots, \xi_N \in \mathbb{R}$. But, $x^\star$ being static means that

$$\dot{x}^\star = \sum_{i=1}^{N} \xi_i \dot{x}_i(0) = \sum_{i=1}^{N} \xi_i \sum_{j \in N_i} (x_j(0) - x_i(0)) = 0,$$

which implies that $\xi_i = \alpha$ for some $\alpha \in \mathbb{R}$. But, since $x^\star$ is static, we also must have that (as $t \to \infty$)

$$x^\star = \sum_{i=1}^{N} \xi_i x^\star = \sum_{i=1}^{N} \alpha x^\star \;\Rightarrow\; \alpha = \frac{1}{N}.$$

And hence

$$x^\star = \frac{1}{N} \sum_{i=1}^{N} x_i(0).$$

Now, consider the weighted consensus equation

$$\dot{x}_i = -\gamma_i \sum_{j \in N_i} w_{ij}(x_i - x_j),$$

where $\gamma_i > 0,\ i = 1, \ldots, N$ and where $w_{ij} = w_{ji} > 0$. We know that the system will still converge to span$\{\mathbf{1}\}$. Use the same type of argument to establish where the system ends up in this case.

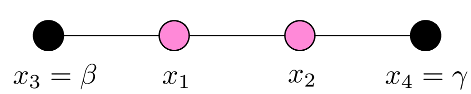
4.

Consider a leader-follower network with two leaders and two followers, as shown below. Assume that the leaders and followers all live on the real line and that the network topology is a line graph, with the edge nodes being the static leaders. Moreover, let the dynamics be given by

$$\begin{aligned}
\dot{x}_1 &= \alpha_1((x_3 - x_1) + (x_2 - x_1)) \\
\dot{x}_2 &= \alpha_2((x_1 - x_2) + (x_4 - x_2)) \\
\dot{x}_3 &= 0 \\
\dot{x}_4 &= 0,
\end{aligned}$$

where $\alpha_1, \alpha_2 > 0$

Where do x_1 and x_2 end up as $t \to \infty$ if $x_3 = \beta$ and $x_4 = \gamma$?



5.

The disagreement vector is given by

$$\delta = \Pi x,$$

where

$$\Pi = I - \frac{1}{N}\mathbf{1}\mathbf{1}^T.$$

Show that $\delta = \Pi x$ is a projection onto span$\{\mathbf{1}\}^\perp$, where $S^\perp$ means the orthogonal compliment to S.

6.

Consider a collection of robots (whose network forms an undirected, connected graph) who are all supposed to meet at one of m possible recharging stations. Assuming that the robots can communicate, explain how you would use the consensus protocol for making the robots agree on a station to recharge at.