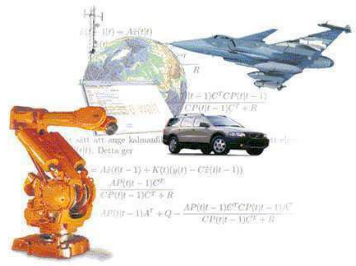
Dynamic Systems

Lecture 1: solutions



Torkel Glad

Reglerteknik, ISY, Linköpings Universitet

The course

- The core of the course is about linear systems in continuous and discrete time.
- Several concepts, e.g. observability, controllability and input-output descriptions are generalized to the nonlinear case.
- The focus is on *analysis*.
- Examination consists of two hand-in-tasks and a take-home exam at the end.

Linear and General Systems

1. Solving system equations
2. Observability
3. Controllability
4. Input-output descriptions
5. Realizability and realizations
6. Canonical forms
7. Matrix fractions
8. DAEs and general linear systems

Linear systems and general systems.

Linear systems:

- Only class of dynamic systems which is (almost) completely understood.
- Only class of dynamic systems where you can do most of the computations you want to.

No physical system is exactly linear
but
most physical systems are approximately linear under suitable conditions (by Taylor expansion of nonlinear differential equations)

Example

Unit mass moving along z-axis with velocity $v \geq 0$, under external force F and aerodynamic drag $-cv^2$

$$\dot{v} = F - cv^2, \quad \dot{z} = v$$

The force $F_0(t)$ gives the solutions $v_0(t)$ and $z_0(t)$. Consider $x_1 = v - v_0$, $x_2 = z - z_0$ and $u = F - F_0$ and linearize the system:

$$\dot{x}_1 = -2cv_0 x_1 + u, \quad \dot{x}_2 = x_1$$

With $a(t) = 2cv_0(t)$

$$\dot{x} = \begin{pmatrix} -a(t) & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

This is a *time-varying* linear system if v_0 is not constant.

When are time-varying linear systems used?

- Linearization around non-equilibrium
- Gain scheduling
- Transients in Kalman filters
- Hybrid systems
- Optimal control, finite horizon
- Periodic phenomena

A general differential equation

$$\dot{x} = f(t, x), \quad x(t_0) = x_0$$

is equivalent to an integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau)) \; d\tau$$

which leads to the iteration

$$x_{j+1}(t) = x_0 + \int_{t_0}^{t} f(\tau, x_j(\tau)) \; d\tau$$

If f is Lipschitz continuous the iteration can be shown to converge to a solution for $t_0 \leq t \leq \epsilon$, provided ϵ is small enough.

The linear case

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0$$

The iteration gives the solution

$$\dot{x}(t) = \Phi(t, t_0)x_0$$

$$\Phi(t, s) = I + \int_s^t A(\sigma)d\sigma + \int_s^t A(\sigma_1) \int_s^{\sigma_1} A(\sigma_2)d\sigma_2 d\sigma_1 + \cdots$$

The *Peano-Baker series* can be shown to converge for arbitrary t and s.
Special case: $A(t) = A$ (constant)

$$\Phi(t, s) = I + (t - s)A + \frac{(t-s)^2}{2}A^2 + \cdots = e^{A(t-s)}$$

Time-varying differential equations in Mathematica

Continued example with $a(t) = 1/(1+t)$ gives

$$\dot{x}_1 = -\frac{x_1}{1+t}, \quad \dot{x}_2 = x_1$$

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DSolve[{x1'[t] == -x1[t] / (1 + t), x2'[t] == x1[t]}, {x1[t], x2[t]}, t]
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{{x1[t] → C[1]/(1 + t), x2[t] → C[2] + C[1] Log[1 + t]}}
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Example, cont'd

$a(t) = 1/(1+t)$ thus gives

$$A(t) = \begin{pmatrix} -\frac{1}{1+t} & 0 \\ 1 & 0 \end{pmatrix}, \quad \Phi(t,s) = \begin{pmatrix} \frac{1+s}{1+t} & 0 \\ (1+s)\log\frac{1+t}{1+s} & 1 \end{pmatrix}$$

while $a(t) = t$ gives

$$A(t) = \begin{pmatrix} -t & 0 \\ 1 & 0 \end{pmatrix}, \quad \Phi(t,s) = \begin{pmatrix} e^{(s^2-t^2)/2} & 0 \\ \sqrt{\frac{\pi}{2}} e^{s^2/2} \left(\text{erf}(\frac{t}{\sqrt{2}}) - \text{erf}(\frac{s}{\sqrt{2}})\right) & 1 \end{pmatrix}$$

Properties of Φ

$$\Phi(t,t) = I$$

$$\frac{d}{dt}\Phi(t,s) = A(t)\Phi(t,s)$$

$$\Phi(t,s) = \Phi(t,\sigma)\Phi(\sigma,s)$$

for all t, s, σ. In particular

$$\Phi(t,s)^{-1} = \Phi(s,t)$$

Linear system solution

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0$$

is given by

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)B(s)u(s)ds$$

If $y(t) = C(t)x(t)$, $x_0 = 0$, then

$$y(t) = \int_{t_0}^{t} h(t,s)u(s)\,ds$$

where the *impulse response* is

$$h(t,s) = C(t)\Phi(t,s)B(s)$$

Discrete time linear systems

$$x(t+1) = A(t)x(t) + B(t)u(t)$$

Forward solution trivial:

$$x(t_0+1) = A(t_0)x(t_0) + B(t_0)u(t_0)$$

$$x(t_0+2) = A(t_0+1)A(t_0)x(t_0) + A(t_0+1)B(t_0)u(t_0) + \\ + B(t_0+1)u(t_0+1)$$

and so on. Compactly written:

$$x(t) = \Phi(t,t_0)x(t_0) + \sum_{j=t_0}^{t-1} \Phi(t,j+1)B(j)u(j)$$

$$\Phi(t,s) = A(t-1)\cdots A(s+1)A(s)$$

The backward solution might not exist or might be non-unique, unless A is invertible.

Carleman linearization

$$\dot{x} = f(x)$$

has an infinite-dimensional linear representation

$$\dot{z} = Az$$

For a system affine in the input:

$$\dot{x} = f(x) + g(x)u$$

there is an infinite dimensional **Carleman bilinearization**

$$\dot{z} = Az + uDz + Bu$$