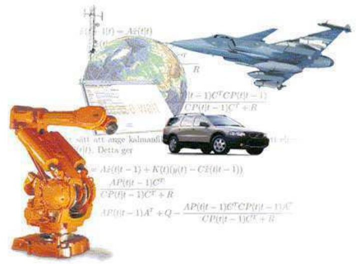
Linear Systems

Lecture 3. Controllability



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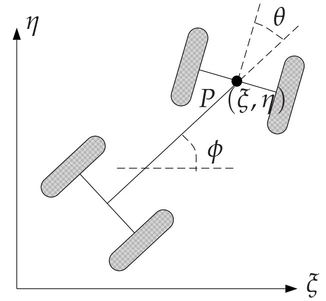
Controllability. Example

Motion planning for a vehicle:



u_1: turning speed of front wheels, u_2: forward speed

$$\dot\theta = u_1, \quad \dot\phi = u_2 \sin\theta$$
$$\dot\xi = u_2\cos(\theta+\phi), \quad \dot\eta = u_2\sin(\theta+\phi)$$

Example, cont'd

In vector form

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \phi\\ \xi\\ \eta\end{bmatrix} = u_1 \underbrace{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}_{f_1} + u_2 \underbrace{\begin{bmatrix}0\\ \sin\theta\\ \cos(\theta+\phi)\\ \sin(\theta+\phi)\end{bmatrix}}_{f_2}$$

How can you span a 4-dimensional space with 2 vectors?

The differential equation solution

$$\dot x = f(x)$$

Taylor expansion:

$$x(h) = x(0) + h\dot x(0) + \frac{h^2}{2}\ddot x(0) + O(h^3)$$

Using $\dot x = f$, $\ddot x = f_x f$

$$x(h) = x(0) + hf(x(0)) + \frac{h^2}{2}(f_x f)(x(0)) + O(h^3)$$

Expanding about $x_o = x(0) + O(h)$

$$x(h) = x(0) + h(f(x_o) + f_x(x_o)(x(0) - x_o)) + \frac{h^2}{2}(f_x f)(x_o) + O(h^3)$$

New directions from Lie brackets

- Start at $x = x_o$.
- Solve $\dot{x} = f_1(x)$ for $t \in [0, h]$, then $\dot{x} = f_2(x)$ for $t \in [h, 2h]$, then $\dot{x} = -f_1(x)$ for $t \in [2h, 3h]$, then $\dot{x} = -f_2(x)$ for $t \in [3h, 4h]$.
- The resulting movement is $h^2[f_1, f_2](x_o) + O(h^3)$
  - $[f_1, f_2] = f_{2,x}f_1 - f_{1,x}f_2$ is the **Lie bracket**.
- By doing nested movements one can generate $[f_3, [f_1, f_2]]$, $[[f_1, f_2], [f_3, f_4]]$, ...
- If the set of all possible Lie brackets spans the space, then intuitively one should have full controllability.

Lie brackets for car example

$$[f_1, f_2] = \begin{bmatrix} 0 \\ \cos\theta \\ -\sin(\theta+\phi) \\ \cos(\theta+\phi) \end{bmatrix}, \quad [f_2, [f_1, f_2]] = \begin{bmatrix} 0 \\ 0 \\ \sin(\phi) \\ -\cos(\phi) \end{bmatrix},$$

Evaluation at $x_o = 0$:

$$f_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad [f_1, f_2] = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad [f_2, [f_1, f_2]] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Properties of Lie bracket

$$[f, g] = g_x f - f_x g$$

Some Lie bracket formulas:

$$[a, a] = 0$$
$$[a, b] = -[b, a]$$
$$[a + b, c] = [a, c] + [b, c]$$
$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0 \quad \text{Jacobi identity}$$

Controllability, definitions

- $A_U(x_0)$: the *reachable set* from x_0, while remaining in the set U
- The system *controllable*: $A_{R^n}(x) = R^n$ for any x.

Problem: If U is a small neighborhood A_U is often "one-sided"
Often the case for systems with drift term:
$\dot{x} = f(x) + u_1 g_1(x) + \cdots + u_m g_m(x)$

The natural local property is local accessibility:

- The system *locally accessible* at x:
  - $A_U(x)$ has a nonempty interior for any neighborhood U of x
  - i.e. $A_U(x)$ has full dimension.

Controllability, the test:

$$\dot{x} = f(x,u)$$

1. form $f_j(x) = f(x,u_j)$, for all possible constant u_j
2. form all possible linear combinations of all possible iterated Lie brackets of the f_j
3. Hermann and Krener 1977: If they span the state space at x_0 (controllability rank condition), then the system is locally accessible at x_0.

Controllability, linear systems

$$\dot{x} = Ax + Bu$$

- $[Ax, b_i] = -Ab_i$, where b_i is a column of B
- $[B, AB, \ldots, A^{n-1}B]$ full rank $\Rightarrow$ controllability rank condition satisfied.

The linear structure permits much stronger conclusions.

To reach with minimum energy

Control from x_0 to x_f with minimum energy:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0, \quad x(t_f) = x_f$$

minimize $J = \frac{1}{2}\int_{t_0}^{t_1} u^T u dt$

The control then has the form

$$u(t) = -B^T(t)\Phi^T(t_0,t)W(t_0,t_1)^{-1}z, \quad z = x_0 - \Phi(t_0,t_1)x_f$$

where W is the *controllability Gramian*

$$W(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_0,t)B(t)B^T(t)\Phi^T(t_0,t)\,dt$$

Controllability

The *controllability Gramian*

$$W(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_0,t)B(t)B^T(t)\Phi^T(t_0,t)\,dt$$

Theorem Possible to move the state from x_0 at $t = t_0$ to x_1 at $t = t_1$
$\Leftrightarrow$
$z = x_0 - \Phi(t_0,t_1)x_1$ is in the range space of $W(t_0,t_1)$.

The energy required is

$$J = \frac{1}{2}z^T W(t_0,t_1)^{-1}z$$

Controllability, time invariant case

Time-invariant case (A, B constant):

$$\mathcal{C} = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}$$

Theorem The range and null spaces of $W(t_0, t)$ coincide with the range and null spaces of $\mathcal{C}\mathcal{C}^T$ for all $t > t_0$.

Discrete time

Iterating the state equation gives

$$R(t_0,t_1)U = z$$

where

$$\begin{aligned} U &= [u^T(t_1-1) \ \ \ldots \ \ u^T(t_0)]^T \\ z &= x(t_1) - A(t_1-1)\cdots A(t_0)x(t_0) \\ R(t_0,t_1) &= [B(t_1-1) \ \ A(t_1-1)B(t_1-2) \ \ \ldots \\ &\quad \ldots \quad A(t_1-1)\cdots A(t_0+1)B(t_0)] \end{aligned}$$

It is possible to move from $x(t_0)$ to $x(t_1)$ in the time interval $[t_0, t_1]$ if and only if z is in the range space of $R(t_0,t_1)$.
The minimum value of U^TU is $z^T(RR^T)^{-1}z$. RR^T thus corresponds to the continuous time controllability Gramian.

Discrete time, time invariant case

$R(t_0,t_1)$ attains maximum rank after at most n time steps (Cayley-Hamilton). The controllability properties are thus given by

$$\mathcal{C} = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} = R(t_0,t_0+n)$$

Controllability theory for time invariant systems is thus the same in continuous and discrete time ($\dot{x}$ replaced by $x(t+1)$).

Change of state variables

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad x = T\bar{x}, \quad T \text{ nonsingular}$$

gives

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u, \quad y = \bar{C}x$$

$$\bar{A} = T^{-1}AT, \ \bar{B} = T^{-1}B, \ \bar{C} = CT, \ \bar{\mathcal{C}} = T^{-1}\mathcal{C}$$

Controllability is thus preserved under this *similarity transformation*.
Theorem Let the rank of $\mathcal{C}$ be r. Then T can be chosen so that

$$\bar{A} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} \bar{B}_{11} \\ 0 \end{pmatrix}$$

where $\bar{A}_{11}$ is $r \times r$, $\bar{B}_{11}$ is $r \times m$ and $\bar{A}_{11}, \bar{B}_{11}$ controllable.
Note: The partition of eigenvalues between $\bar{A}_{11}$ and $\bar{A}_{22}$ is unique: “controllable” and “uncontrollable” eigenvalues

The Kalman decomposition theorem

R range space of controllability matrix; N null space of observability matrix

$$T = \begin{bmatrix} T_1 & T_2 & T_3 & T_4 \end{bmatrix}$$

where T_2 basis for $R \cap N$, $[T_2, T_4]$ basis for N, $[T_1, T_2]$ basis for R. Then

$$\dot{z} = \begin{bmatrix} \tilde{A}_{11} & 0 & \tilde{A}_{13} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} \\ 0 & 0 & \tilde{A}_{33} & 0 \\ 0 & 0 & \tilde{A}_{43} & \tilde{A}_{44} \end{bmatrix} z + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \tilde{C}_1 & 0 & \tilde{C}_3 & 0 \end{bmatrix}$$

i.e. system decomposed into controllable-observable, controllable-unobservable, uncontrollable-observable and uncontrollable-unobservable parts.

PBH tests

Theorem A, B controllable if and only if

$$p^T A = \lambda p^T, \; p^T B = 0 \Rightarrow p = 0$$

Theorem A, B controllable if and only if

$$\text{rank} \begin{pmatrix} sI - A & B \end{pmatrix} = n$$

for all complex s.

Nonlinear stabilizability

$$\dot{x} = f(x, u), \quad f(x_o, u_o) = 0$$

Is it possible to find a feedback that gives asymptotic stability at x_o?

- Linearization around x_o, u_o has all “uncontrollable” eigenvalues strictly in left half plane $\Rightarrow$ possible.
- Linearization around x_o, u_o has an “uncontrollable” eigenvalue strictly in right half plane $\Rightarrow$ impossible.
- Difficult case: “uncontrolable” eigenvalues on imaginary axis.