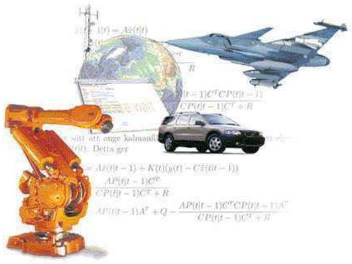
Nonlinear control

Lecture 3: Observers



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The course so far

- Input-output linearization
  - Requires nice zero dynamics
  - Gives both decoupling and desired input output dynamics
  - “Cost of control” not addressed
- Transformation to controller form
  - No zero dynamics
  - Restrictive Lie bracket conditions
  - Arbitrary transformation of dynamics possible
  - Computational issues

State feedback required.

Linear observers – a reminder

$$\begin{aligned} \text{System:}\quad & \dot{x} = Ax + Bu, \quad y = Cx \\ \text{Observer:}\quad & \dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}) \\ \text{Error dynamics:}\quad & \dot{\tilde{x}} = (A - KC)\tilde{x}, \quad \tilde{x} = x - \hat{x} \end{aligned}$$

- A, C observable $\Rightarrow K$ can be chosen to give $A - KC$ any desired eigenvalues.
- Stochastic disturbance model leads to Kalman filter.

Two nonlinear observer approaches

- Nonlinear filtering – focus on stochastic disturbance model
  - **Extended Kalman filter**. Linearize around $\hat{x}$ at each time instant and use ordinary Kalman filter.
  - Compute conditional probability exactly: computationally intractable (nonlinear PDEs in real time).
  - Approximations, e.g.**particle filters**
- Nonlinear observers – focus on stability and convergence
  - High-gain observers
  - Transformation to get linear error dynamics
  - Differentiation and equation solving

SISO high gain observers: canonical form

$$\dot{x} = f(x) + \bar{g}(x)u, \quad y = h(x)$$

Introduce new variables:

$$z_1 = h(x), \quad z_2 = (L_f h)(x), \ldots, z_n = (L_f^{n-1} h)(x)$$

gives the canonical form

$$\dot{z} = \begin{bmatrix} z_2 \\ \vdots \\ z_n \\ \phi(z) \end{bmatrix} + \begin{bmatrix} g_1(z) \\ \vdots \\ g_{n-1}(z) \\ g_n(z) \end{bmatrix} u$$

SISO high gain observer: basic set-up

The canonical form can be written

$$\dot{z} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & 0 \\ \vdots & & & & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}}_{A} z + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{B} \phi(z) + g(z)u, \quad y = \underbrace{\begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}}_{C} z$$

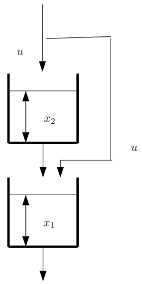
with the observer

$$\dot{\hat{z}} = A\hat{z} + B\phi(\hat{z}) + g(\hat{z})u + K(y - C\hat{z})$$

and the observer error $\tilde{z} = z - \hat{z}$

$$\dot{\tilde{z}} = (A - KC)\tilde{z} + B(\phi(z) - \phi(\hat{z})) + (g(z) - g(\hat{z}))\, u$$

SISO high gain observers. An example



$$\begin{aligned} \dot{x}_1 &= -\sqrt{x_1} + \sqrt{x_2} + u \\ \dot{x}_2 &= -\sqrt{x_2} + u \\ y &= x_1 \end{aligned}$$

With $z_1 = y = x_1, \quad z_2 = L_f x_1 = -\sqrt{x_1} + \sqrt{x_2}$ the system is

$$\dot{z} = \begin{bmatrix} z_2 \\ \phi(z) \end{bmatrix} + \begin{bmatrix} 1 \\ g_2(z) \end{bmatrix} u$$

SISO high gain observers. Example, cont'd

The system dynamics

$$\dot{z} = \underbrace{\begin{bmatrix} o & 1 \\ 0 & 0 \end{bmatrix}}_{A} z + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} \phi(z) + \underbrace{\begin{bmatrix} 1 \\ g_2(z) \end{bmatrix}}_{g(z)} u, \quad y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} z$$

gives the error dynamics

$$\dot{\tilde{z}} = (A - KC)\tilde{z} + B \underbrace{\left((\phi(z) - \phi(\hat{z})) + (g_2(z) - g_2(\hat{z}))u\right)}_{\delta}$$

The transfer function from δ to $\tilde{z}$ is

$$(sI - A + KC)^{-1}B = \frac{1}{s^2 + k_1 s + k_2} \begin{bmatrix} 1 \\ s + k_1 \end{bmatrix}$$

whose gain can be made arbitrarily small.

SISO high gain observers. Summary

- Gives a low gain to the transfer function from nonlinearity to observer error by using large values of K.
- Complete stability analysis requires Lyapunov theory (Lecture 5)
- The obvious drawback is the sensitivity to measurement errors.

Another useful canonical form (SISO)

The observer canonical form makes it easy to construct an observer.

$$\dot{x} = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ -a_{n-1} & 0 & \dots & 0 & 1 \\ -a_n & 0 & \dots & 0 & 0 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} x$$

An analogous nonlinear observer form is

$$\begin{aligned} \dot{x}_1 &= f_1(u, y) + x_2 \\ \dot{x}_2 &= f_2(u, y) + x_3 \\ &\vdots \\ \dot{x}_{n-1} &= f_{n-1}(u, y) + x_n \\ \dot{x}_n &= f_n(u, y) \end{aligned}$$

Transformation to get linear error dynamics

If the system has the form (e.g. nonlinear observer form)

$$\dot{x} = Ax + f(u, y), \quad y = Cx$$

then the observer

$$\dot{\hat{x}} = A\hat{x} + f(u, y) + K(y - C\hat{x})$$

has the linear error dynamics

$$\dot{\tilde{x}} = (A - KC)\tilde{x}$$

State transformation to get linear error dynamics

Can you transform

$$\dot{x} = f(x), \quad y = h(x)$$

into the form

$$\begin{aligned} \dot{\xi} &= \tilde{f}(\xi), & y &= \tilde{h}(\xi) \\ \tilde{f}(\xi) &= A\xi + b(y), & \tilde{h}(\xi) &= C\xi \end{aligned}$$

using a coordinate change $x = X(\xi)$?
Then we must have

$$f(X(\xi)) = J(\xi)\tilde{f}(\xi), \quad J \text{ Jacobian of } X$$

SISO conditions for linear error dynamics

The columns J_k of J are calculated from

$$\underbrace{\begin{bmatrix} h_x \\ (L_f h)_x \\ \vdots \\ (L_f^{\sigma-1} h)_x \end{bmatrix}}_{Q} J_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad J_{k-1} = -[f, J_k],\ k = n, \dots, 2$$

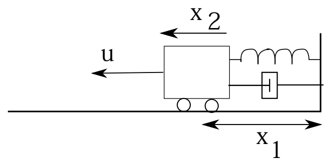
The transformation problem is solvable iff

- Q is nonsingular
- $J^{-1}(x)$ integrable

The solution is obtained by calculating

$$\xi = Y(x), \quad \text{where } Y_x(x) = J^{-1}(x)$$

Linear observer error dynamics. An example



A mechanical system with nonlinear spring and position-dependant damping.

$$\dot{x}_1 = x_2 \qquad \dot{\xi}_1 = b_1(\xi_1, u) + \xi_2$$
$$\dot{x}_2 = -k(x_1) - \sigma(x_1)x_2 + u \qquad \dot{\xi}_2 = b_2(\xi_1, u)$$
$$y = x_1 \qquad y = \xi_1$$

Find the transformation

Linear observer error dynamics. Example – solution

$$\xi_1 = x_1$$
$$\xi_2 = \Sigma(x_1) + x_2, \quad \Sigma(x_1) = \int^{x_1} \sigma(v)dv$$

$$\dot{\xi}_1 = -\Sigma(\xi_1) + \xi_2$$
$$\dot{\xi}_2 = -k(\xi_1) + u$$

Multi-output observers. Observability structure

For $\dot{x} = f(x,u)$, $y = h(x)$ differentiate outputs so that the following equations define x uniquely, given u and y

$$y_1 = h_1, \quad \dot{y}_1 = L_f h_1, \quad \dots \quad y^{(\sigma_1-1)} = L_f^{\sigma_1-1} h_1$$
$$\vdots\vdots$$
$$y_p = h_p, \quad \dot{y}_p = L_f h_p, \quad \dots \quad y^{(\sigma_p-1)} = L_f^{\sigma_p-1} h_p$$

Label the outputs so that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p$.

Multi-output observers. Desired form

The desired description with p outputs is

$$\dot{\xi} = A\xi + b(y), \quad y = C\xi$$

$$A = \begin{bmatrix} 0_{p\times p} & I_{p\times\ell_1} & 0 & \dots & \\ 0_{\ell_1\times p} & 0_{\ell_\times\ell_1} & I_{\ell_1\times\ell_2} & & \\ & & & \ddots & \\ & & & & I_{\ell_{\sigma-1}\times\ell_\sigma} \\ 0_{\ell_\sigma\times p} & & & & 0_{\ell_\sigma\times\ell_\sigma} \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 0_{p\times\ell_1} & \dots & 0_{p\times\ell_\sigma} \end{bmatrix}$$

where C_1 is $p \times p$, ℓ_i is the number of outputs that are differentiated i times, $\sigma =$ highest derivative $+1$.

Try $C_1 = I$ first.

Multi-output. Conditions on transformation

Lie bracket conditions

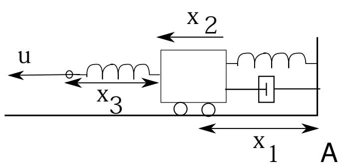
$$[f, J_k] = -Jf_{\xi,k}$$

where index k denotes the $k : th$ column.

Lie derivative conditions

$$\begin{bmatrix} h_x \\ (L_f h)_x \\ \vdots \\ (L_f^{\sigma-1}h)_x \end{bmatrix} J = \begin{bmatrix} C_1 I_p & 0 & 0 & \dots & 0 \\ * & C_1 I_{p\times\ell_1} & 0 & \dots & 0 \\ \vdots & & \ddots & & \\ * & * & \dots & & C_1 I_{p\times\sigma-1} \end{bmatrix}$$

Multi-output observer example



A mechanical system with two nonlinear springs and position-dependant damping.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k(x_1) - \sigma(x_1)x_2 + \ell(x_3) \\ \dot{x}_3 &= -x_2 + u \\ y_1 &= x_1 \\ y_2 &= x_3 \end{aligned}$$

$$\begin{aligned} \dot{\xi}_1 &= b_1(\xi_1, \xi_2, u) + \xi_2 \\ \dot{\xi}_2 &= b_2(\xi_1, \xi_2, u) \\ \dot{\xi}_3 &= b_3(\xi_1, \xi_2, u) \\ y_1 &= \xi_1 \\ y_2 &= c\xi_1 + \xi_2 \end{aligned}$$

Find the transformation

Multi-output observer example – solution

$$\begin{aligned} \xi_1 &= x_1 \\ \xi_2 &= x_1 + x_3 \\ \xi_3 &= \Sigma(x_1) + x_2, \quad \Sigma(x_1) = \int^{x_1} \sigma(v)dv \end{aligned}$$

$$\begin{aligned} \dot{\xi}_1 &= -\Sigma(\xi_1) + \xi_3 \\ \dot{\xi}_2 &= u \\ \dot{\xi}_3 &= -k(\xi_1) + \ell(\xi_2 - \xi_1) \end{aligned}$$

Observers with linear error dynamics – extensions

The transformations can only be carried out under very restrictive conditions. There are however many extensions in the litterature:

- Allow nonlinear transformations of the output. (Covered in the lecture notes).
- Allow transformation of the time variable.
- Embed the dynamics an a larger state space.
- ....

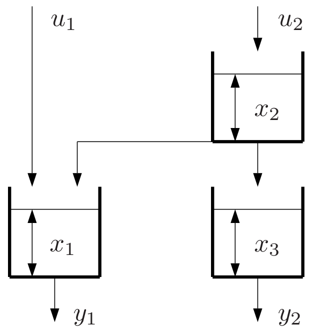
Differentiation and equation solving

Idea: Compute x from the nonlinear system of equations

$$y_1 = h_1, \quad \dot{y}_1 = L_f h_1, \quad \ldots \quad y^{(\sigma_1 - 1)} = L_f^{\sigma_1 - 1} h_1$$
$$\vdots$$
$$y_p = h_p, \quad \dot{y}_p = L_f h_p, \quad \ldots \quad y^{(\sigma_p - 1)} = L_f^{\sigma_p - 1} h_p$$

- How do you get $\dot{y}_i$, $\ddot{y}_i$ etc.? – Use approximate differentiation e.g. $\frac{1}{1+sT}$
- How do you solve the nonlinear equations? Problem dependent.

A simple example



A model is

$$\dot{x}_1 = -\sqrt{x_1} + \sqrt{x_2} + u_1$$
$$\dot{x}_2 = -2\sqrt{x_2} + u_2$$
$$\dot{x}_3 = -\sqrt{x_3} + \sqrt{x_2}$$
$$y_1 = x_1$$
$$y_2 = x_3$$

$\sigma_1 = 2, \sigma_2 = 1$ ($\sigma_1 = 1, \sigma_2 = 2$ also works)

$$x_1 = y_1, \quad x_2 = (\dot{y}_1 + \sqrt{y_1} - u)^2, \quad x_3 = y_2$$

Separation theorems

Separation theorem: Good state feedback + good observer gives good overall performance.

- Linear systems:
  - Pole placement of state feedback + pole placement of observer gives overall pole placement
  - LQG: optimal feedback + Kalman filter gives overall optimality
- Nonlinear systems: few and very restrictive results.