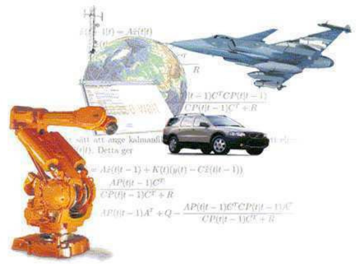
Nonlinear control

Lecture 5. Lyapunov based design



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Review of course

- Geometric control theory
  - input-output linearization
  - controller canonical form
  - observer canonical form
- Lyapunov theory
  - Stability results
  - Passivity
  - Circle and Popov criteria

Lyapunov design

1. Control Lyapunov functions
2. Back-stepping
3. Forwarding
4. Observers
5. (Disturbance supression)
6. (Passivity based control)

Control Lyapunov functions

V is a **Control Lyapunov function** if

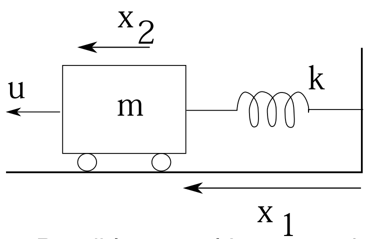
for every x there is some u so that $V_x f(x,u) < 0$

Choose $u = k(x)$ so that

$$V_x f(x, k(x)) \text{ negative definite}$$

Might be difficult to find nice k.

Control Lyapunov function. Example



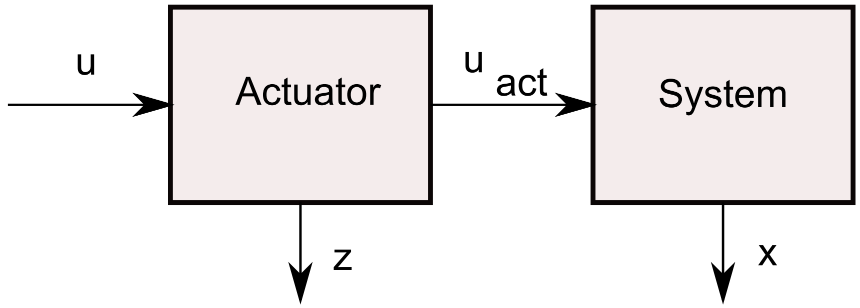
The spring k has cubic stiffness.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1^3 + u$$

Possible control Lyapunov functions:

$$V_1 = \underbrace{\frac{x_1^4}{4}}_{\text{potential energy}} + \underbrace{\frac{x_2^2}{2}}_{\text{kinetic energy}}, \quad V_2 = \frac{x_1^2}{2} + \frac{x_1^4}{4} + \frac{x_2^2}{2}$$

A typical block structure



- Design a controller assuming u_{act} is the control signal
- Step back to the real control signal u and extend the controller design (“backstepping”).

Back-stepping with Lyapunov functions

If the actuator is first order one can take $u_{act} = z$:

$$\dot{x} = f(x) + g(x)z$$
$$\dot{z} = a(x,z) + b(x,z)u$$

Suppose we find a control law $z = k(x)$ and a Lyapunov function $V(x)$ so that

$$V_x(x)(f(x) + g(x)k(x)) = -W(x), \quad V, W \text{ positive definite}$$

This control law can then be extended (the “step back”) to a control law for u, using a control Lyapunov function, e.g.

$$V_e(x,z) = V(x) + \frac{1}{2}(z - k(x))^2$$

Repeated backstepping

Systems in feedback form:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1,x_2) + g_2(x_1,x_2)x_3 \\ &\vdots \\ \dot{x}_n &= f_n(x_1,\dots,x_n) + g_n(x_1,\dots,x_n)u \end{aligned} \qquad \begin{aligned} \dot{x}_1 &= f_1(x_1,x_2) \\ \dot{x}_2 &= f_2(x_1,x_2,x_3) \\ &\vdots \\ \dot{x}_n &= f_n(x_1,\dots,x_n,u) \end{aligned}$$

Repeated backstepping is easily done for the structure to the left. It can also be generalized to the structure to the right.

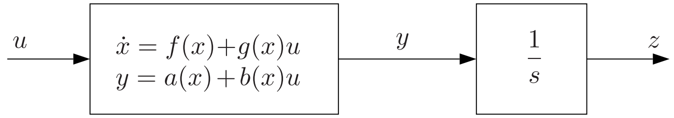
Back-stepping

Basic advantage: Not necessary to cancel terms that make $\dot{V}_e$ negative. Many opportunities for creative extensions.

Tutorial, theory and applications in:
Ola Härkegård: Back-stepping and Control Allocation with Applications to Flight Control. *PhD. Thesis, Department of Electrical Engineering, Linköping University, 2003*

Forwarding

Extending the Lyapunov function when an integrator is added to the output.



Suppose positive definite functions V, W and a control law k are known so that

$$V_x(x)(f(x) + g(x)k(x)) = -W(x)$$

Let $\zeta = \phi(x,z)$ be constant when $u = k(x)$. Then

$$V_e(x,z) = V(x) + \frac{1}{2}\zeta^2$$

is a suitable control Lyapunov function.

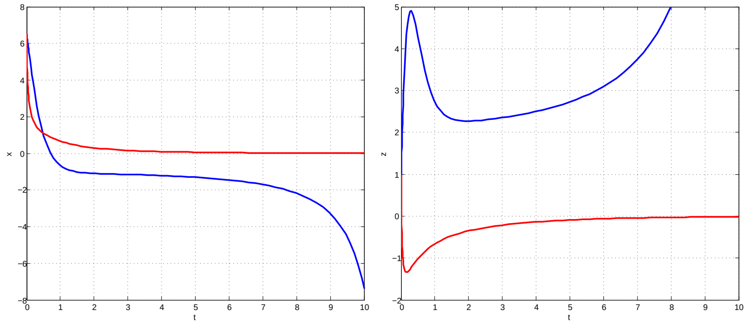
Forwarding example

$$\dot{z} = x^2 + u$$
$$\dot{x} = -x + u$$

Stabilization of x-system: $u = k(x) = 0$.
Coordinate change: $\zeta = z + \frac{x^2}{2}$
Resulting control law:
$u = -x - (1+x)\zeta$

Result, forwarding example

Forwarding control for: $x(0) = 6.5$, $z(0) = 1$



Left diagram: x, right diagram: z
full forwarding controller: red
linear part of the controller: blue

The high gain observer

$$\dot{x} = Ax + B\phi(x) + g(x)u, \quad y = Cx$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & 0 \\ \vdots & & & & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$$

The observer is

$$\dot{\hat{x}} = A\hat{x} + B\phi(\hat{x}) + g(\hat{x})u + K(y - C\hat{x})$$

A Lyapunov function for the observer

The observer error is

$$\dot{\tilde{x}} = (A - KC)\tilde{x} + \underbrace{B(\phi(x) - \phi(\hat{x})) + (g(x) - g(\hat{x}))\,u}_{L(x,\hat{x},u)}$$

With $K = S^{-1}C^T$ and S given by

$$A^T S + SA - C^T C = -\theta S$$

the function
$V = \tilde{x}^T S \tilde{x}$
is a Lyapunov function, if θ is large enough.

State feedback via observer

State feedback: $\dot{x} = f(x) + g(x)u$, $u = k(x)$
Lyapunov function V:

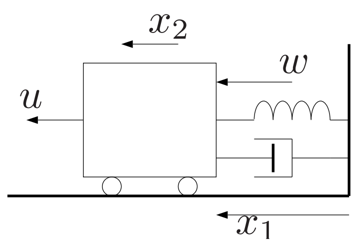
$$\dot{V} = V_x(x)(f(x) + g(x)k(x)) \leq -q(x) \leq 0$$

Observer: observer error $\tilde{x} = x - \hat{x}$
Lyapunov function $V_e = |\tilde{x}|_Q^2$ for some norm $||_Q$:

$$\dot{V}_e = \leq -q_e(x) \leq 0$$

Is $W(x, \tilde{x}) = V(x) + V_e(\tilde{x})$ a Lyapunov function for the closed loop system?

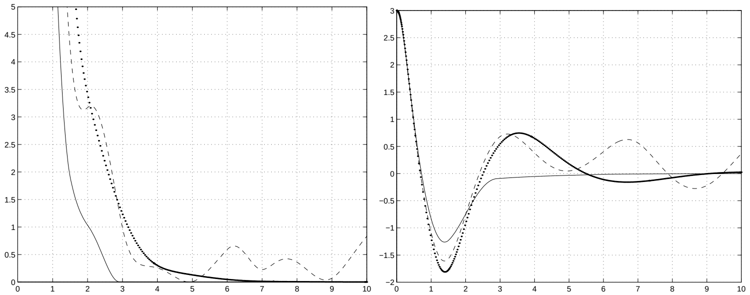
Disturbances



$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_1^3 - x_2 + u + w$$

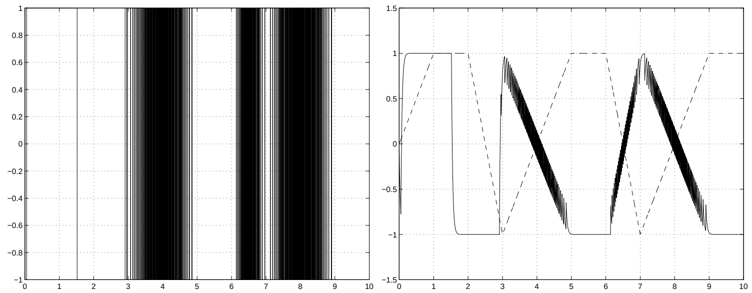
Disturbance suppression

Result of disturbance compensation



To the left: $V(t)$, to the right: $x_1(t)$.
Uncontrolled system without disturbance: dotted,
uncontrolled system with disturbance: dashed,
controlled system with disturbance: solid.

Disturbance example



Disturbance w, control signal u, and filtered control signal

Passivity

Passive system:

$$\int_0^T u^T y \, dt + \gamma(x(0)) \geq 0$$

Design a feedback law

$$u = v - k(x)$$

so that the system is still passive from v to y.

Passivity and feedback

If two passive systems are connected in a feedback loop, the resulting system is passive: