A Critical View on Benchmarks based on Randomly Generated Systems

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- It is now customary to rely on data sets from randomly generated systems
- Here we discuss the implications of this practice, in particular when using data sets generated with the MATLAB® command `drss`

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- Cheap to generate random systems!

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The MATLAB® command drss generates random discrete-time linear systems in state-space form

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3. B, C and D are generated

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- Arguments of each conjugate pair: $\pm\mathcal{U}[0, \pi]$

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$$\begin{bmatrix} \mathrm{Re}(p_i) & \mathrm{Im}(p_i) \\ -\mathrm{Im}(p_i) & \mathrm{Re}(p_i) \end{bmatrix}$$

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- $U \in \mathbb{R}^{n \times n}$: orthogonalization of $n \times n$ $\mathcal{U}[0, 1]$ matrix

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Generation of B, C and D

$B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$ and $D \in \mathbb{R}^{1\times 1}$: $\mathcal{N}(0,1)$ random matrices

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Remark In addition, `drss` zeroes some entries of B, C, D with prescribed probability

Properties of systems generated by `drss`

(i) Benchmarks of random systems induce a Bayesian comparison of identification techniques

(ii) Poles of the systems generated by `drss` do not reflect standard sampling rules-of-thumb

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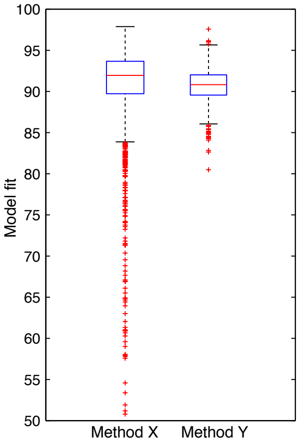

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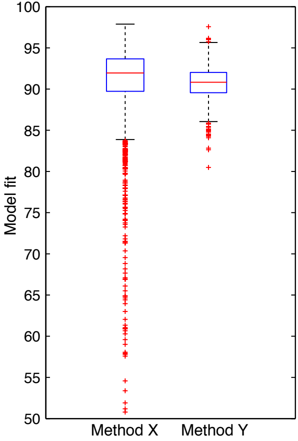

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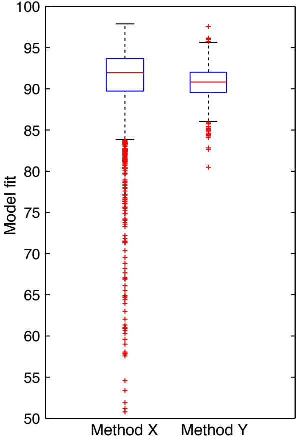

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It depends on the distribution of the systems!

A Bayesian prior on linear systems (cont.)

Distribution of the model fit

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Is it a natural (non-informative) or realistic prior?

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Poles of generated systems

If a `drss`-generated system has n' distinct poles, their maximum magnitude is close to 1 with high probability for large n'

(the expected maximum magnitude is $n'/(n'+1)$)

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- $\Rightarrow$ `drss` can generate, for large n, the equivalents of severely over-sampled systems

Poles of generated systems (cont.)

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- Similarly, random systems with dominant poles of small magnitude correspond to under-sampled systems, whose estimation can be difficult (poor observability/identifiability)

- In summary: **poles of random systems should be carefully placed to represent how sampled systems would look like** (assuming sampling and experiment design are properly done)

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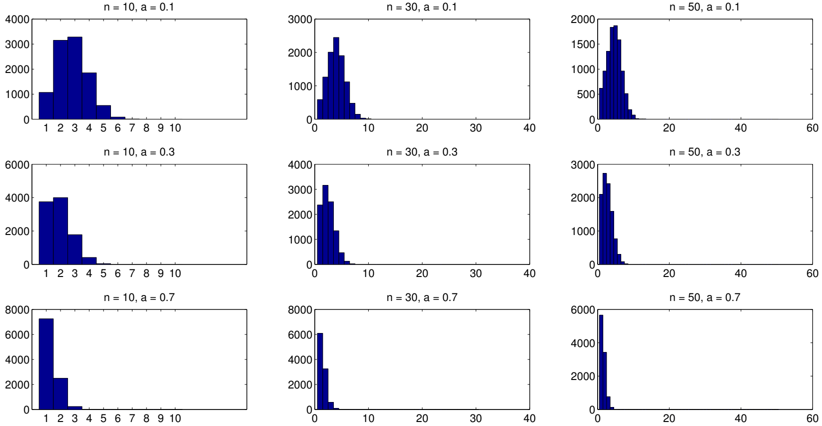
where
$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$: Hankel singular values of G
a: threshold on number of significant Hankel singular values

Low order random systems (cont.)



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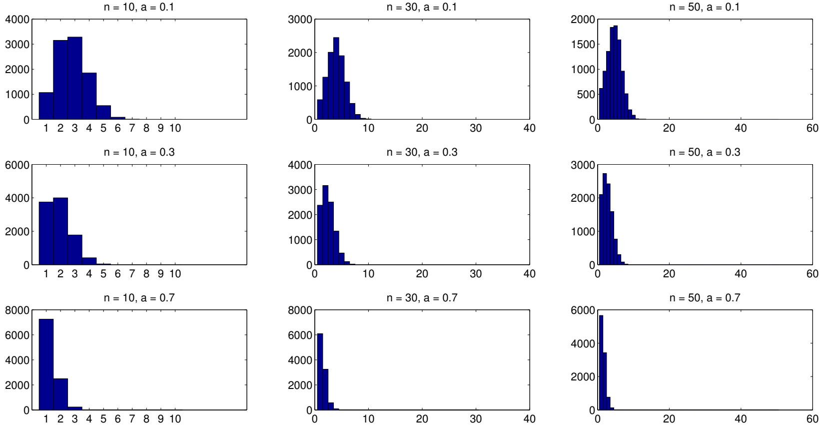

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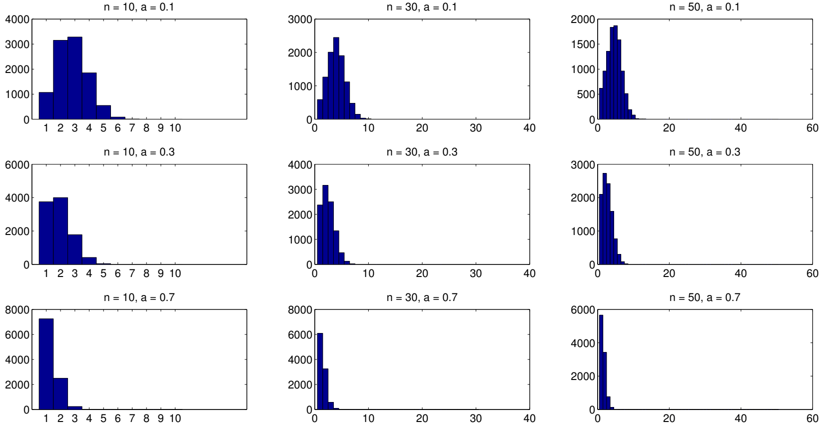

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is this good or bad?

Some suggestions

- Partition the benchmark systems into subsets
- Plot the joint distribution of performance measures
- Sample randomly generated continuous-time systems
- Try "irreducible" systems

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- An alternative is to split the set of generated systems into several subclasses (with similar dynamics, order, resonances, *etc.*), and test the estimators on each subclass *separately*
- This would allow to distinguish conditions under which an estimator outperforms others

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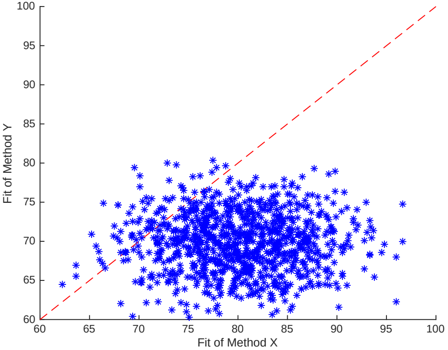

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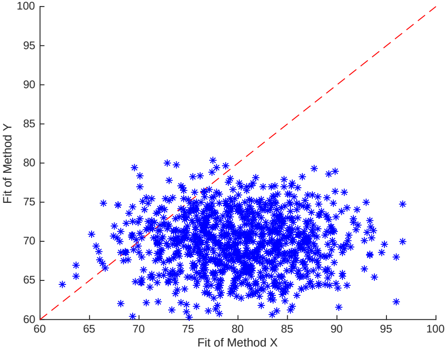

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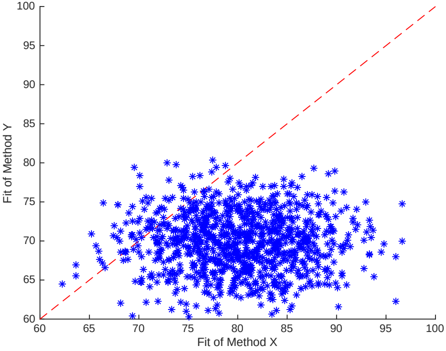

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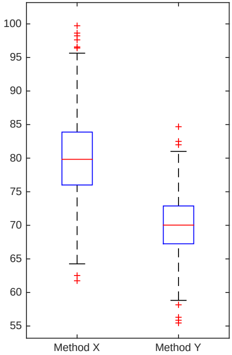

Sample random continuous-time systems

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- Which prior to use in continuous-time?

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- These systems may not be *realistic*, but may serve to test estimators on problem of real high order

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  - How should we present the results of Monte Carlo studies?

Thank you!