Chapter 11

Nonlinear feedforward control

11.1 Basic ideas of feedforward control

A basic control problem is to generate a control signal u so that the output y of a physical system follows a given reference signal r. The simplest configuration is shown in figure 11.1, where S is the controlled system and F is the controller. If F and S are regarded as operators for computing the output from the input,

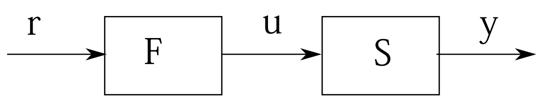

Figure 11.1: Feedforward from the reference signal.

then we want

$$y = S(Fr) \Rightarrow y = r \tag{11.1}$$

which implies that F should be the inverse of S,

$$F = S^{-1} \tag{11.2}$$

Now consider a situation where the objective is to reject a disturbance. In figure 11.2 the output y is the sum of the output from the system S, having input u and the output from system S_d having input d. The objective is to keep the output zero despite the disturbance d. From the relation

$$y = (S_d + SF)d$$

it followa that F should be chosen as

$$F = -S^{-1}S_d$$

These two examples show that in order to do feedforward, either from a reference signal or a disturbance one has to use an inverse of the physical system. The computation of a system inverse is thus a crucial step in feedforward design.

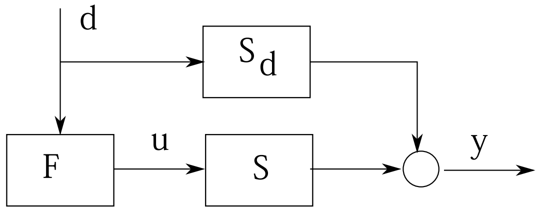

Figure 11.2: Feedforward from a disturbance.

11.2 Linear feedforward

Consider equation (11.1) for single-input-single-output linear systems. Let $S = b(s)/a(s)$ where b and a are polynomials in the Laplace variable s. Then from (11.2) $F = a(s)/b(s)$, so that we get

$$y = \frac{b(s)}{a(s)} \frac{a(s)}{b(s)} r = r \tag{11.3}$$

This feedforward controller thus achieves the desired purpose. There are however two difficulties.

(I) The transfer function calculations in (11.3) assume that the initial conditions of both systems are zero. Otherwise initial transients should be added to the expression for y.

- If both $a(s)$ and $b(s)$ have all their roots strictly in the left half plane, then all initial transients go to zero exponentially. Therefore $y = r$ will be satisfied asymptotically.
- If either $a(s)$ or $b(s)$ has a root in the right half plane the series connection is an unstable system and the feedforward does not work.

(II) If the transfer function of the physical system is strictly proper the inverse will be non-proper, i.e. the degree of the numerator will be greater than the degree of the denominator. This means that the inverse will act as a differentiator giving problems when dealing with real physical signals.

11.3 Nonlinear feedforward

Now consider the case of a nonlinear system

$$\dot{x} = f(x) + g(x)u, \quad y = h(x) + D(x)u \tag{11.4}$$

Note that there is a direct term $D(x)u$ relating the input to the output ($D = 0$ in most applications). It is assumed that $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$ (i.e. same number of inputs and outputs) and that $x(t) \in \mathbb{R}^n$.

Nonsingular D

In the case where D in (11.4) is nonsingular it turns out to be easy to invert the system. This is because (11.4) can be rearranged by solving for u in the output equation and then substituting into the state equation. This gives

$$\dot{x} = f(x) + g(x)D(x)^{-1}(-h(x) + y), \quad u = D(x)^{-1}(-h(x) + y) \tag{11.5}$$

With the differential equation written in this way it is natural to regard y as the input and u as the output, so that this is a state space representation of the inverse. Using $\hat{x}$ for the state of the inverse system one can then write the series connection (11.1) as

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}) + g(\hat{x})D(\hat{x})^{-1}(-h(\hat{x}) + r) \\ \dot{x} &= f(x) + g(x)D(\hat{x})^{-1}(-h(\hat{x}) + r) \\ y &= h(x) + D(x)D(\hat{x})^{-1}(-h(\hat{x}) + r) \end{aligned} \tag{11.6}$$

The basic property is in the following proposition.

Proposition 11.1 Suppose the system (11.6) is initialized so that $\hat{x}(0) = x(0)$. Then for all positive t it holds that $\hat{x}(t) = x(t)$ and $y(t) = r(t)$.

Proof. Substituting into the differential equation it is immediately clear that $\hat{x}(t) = x(t)$ is a possible solution if the initial condition is $\hat{x}(0) = x(0)$. Since the solution is unique the result follows. ■

The proposition shows that difficulty (I) from the linear case is still present. The inverse is only exact when the initial conditions are correct. Otherwise one has to rely on some sort of stability result. Sometimes Lyapunov theory helps, as in the following example.

Example 11.1 Consider the electric circuit shown in figure 11.3. The input

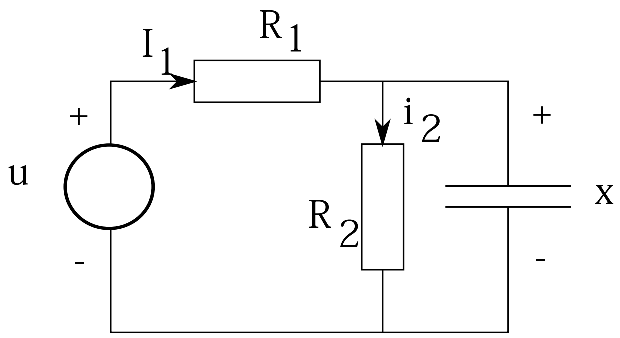

Figure 11.3: Nonlinear electric circuit.

u is the voltage of an ideal voltage source. The state is the voltage of the capacitor. The resistor R_1 is linear with unit resistance, while R_2 is nonlinear with the relation $I_2 = g(x)$. Then

$$\dot{x} = I_1 - I_2, \quad I_1 = u - x, \quad I_2 = g(x)$$

and the complete model is

$$\dot{x} = -x - g(x) + u \tag{11.7}$$
$$y = -x + u \tag{11.8}$$

The inverse then becomes

$$\dot{\hat{x}} = -g(\hat{x}) + r \tag{11.9}$$
$$u = \hat{x} + r \tag{11.10}$$

and the series connection of the two systems is

$$\dot{\hat{x}} = -g(\hat{x}) + r \tag{11.11}$$
$$\dot{x} = -x - g(x) + \hat{x} + r \tag{11.12}$$
$$y = -x + \hat{x} + r \tag{11.13}$$

A simulation of this series connection is shown in figure 11.4 for the case $g(x) = x^3$. The initialization is $x(0) = 3$ and $\hat{x}(0) = 0$. It can be seen that the output

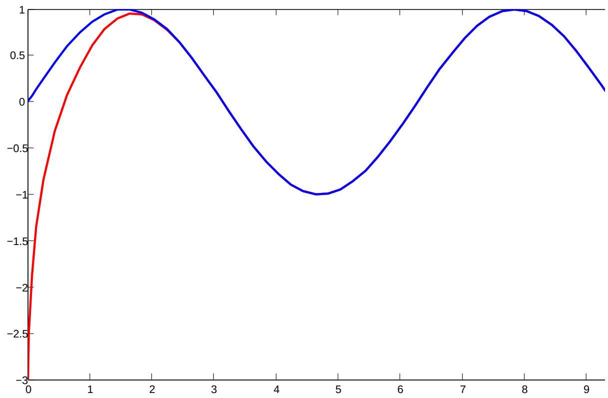

Figure 11.4: Series connection of circuit model and its inverse. Reference signal starts at 0 and output at −3.

rapidly converges to the reference. The convergence of $\hat{x}$ to x can also be seen by using the Lyaounov function $V = \frac{1}{2}(x - \hat{x})^2$.

$$\dot{V} = (x - \hat{x})(-(x - \hat{x}) - (x^3 - \hat{x}^3)) \leq -(x - \hat{x})^2$$

■

$D = 0$ but nonsingular decoupling matrix

If the system is

$$\dot{x} = f(x) + g(x)u, \quad y = h(x) \tag{11.14}$$

i.e. $D = 0$ in (11.4) one can use the technique from section 3.1. It was shown that

$$Y = d(x) + R(x)u \tag{11.15}$$

where Y, d and R are given by

$$Y = \begin{pmatrix} y_1^{(\nu_1)} \\ \vdots \\ y_m^{(\nu_m)} \end{pmatrix}, \quad d(x) = \begin{pmatrix} L_f^{\nu_1} h(x) \\ \vdots \\ L_f^{\nu_m} h(x) \end{pmatrix}, \quad R(x) = \begin{pmatrix} L_{g_1} L_f^{\nu_1 - 1} h_1 & \dots & L_{g_m} L_f^{\nu_1 - 1} h_1 \\ \vdots & & \vdots \\ L_{g_1} L_f^{\nu_m - 1} h_m & \dots & L_{g_m} L_f^{\nu_m - 1} h_m \end{pmatrix} \tag{11.16}$$

The system (11.14) can now be replaced by

$$\dot{x} = f(x) + g(x)u, \quad Y = d(x) + R(x)u \tag{11.17}$$

If the decoupling matrix R is nonsingular the system can be rewritten in analogy with (11.5) to give the inverse

$$\dot{\hat{x}} = f(\hat{x}) + g(\hat{x})R(\hat{x})^{-1}(-d(\hat{x}) + Y) \tag{11.18}$$

$$u = R(\hat{x})^{-1}(-d(\hat{x}) + Y) \tag{11.19}$$

Here difficulty (II) from the linear case is seen in the nonlinear context: Y contains differentiations of the output.

The Silverman-Hirschorn-Singh algorithm

If the decoupling matrix R turns out to be singular or if $D(x)$ in (11.4) is nonzero but singular the computation of an inverse turns out to be more complicated. This is because further differentiation of the output now would lead to differentiation of u. Suppose the system is given by

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) + D(x)u \end{aligned} \tag{11.20}$$

where $D(x)$ is singular (possibly $D = 0$). Let the rank of $D(x)$ be $r < m$ in some open set Ω. By permuting rows one can then make the first r rows in D linearly independent. Let this be achieved by multiplication with the matrix E_1 from the left. Then

$$E_1 y = E_1 h(x) + \begin{bmatrix} D_a(x) \\ D_b(x) \end{bmatrix} u \tag{11.21}$$

where D_a is an $r \times m$ matrix of rank r and all rows in D_b are linear combinations of rows in D_a. This means that D_b can be zeroed by subtraction of suitable linear combinations of rows in D_a or put in another way: there exists a matrix $F(x)$ so that $D_b(x) + F(x)D_a(x) = 0$. Now multiply from the left with the matrix

$$E_2(x) = \begin{bmatrix} I_r & 0 \\ F(x) & I_{m-r} \end{bmatrix} \tag{11.22}$$

Then

$$E_2(x)E_1 y = E_2(x)E_1 h(x) + \begin{bmatrix} D_a(x) \\ 0 \end{bmatrix} u \tag{11.23}$$

Define

$$\tilde{y} = E_1 y = \begin{bmatrix} y_a \\ y_b \end{bmatrix}, \quad E_1 h = \begin{bmatrix} h_a \\ h_b \end{bmatrix}$$

Then (11.23) can be written

$$\begin{bmatrix} y_a \\ y_b + F(x)y_a \end{bmatrix} = \begin{bmatrix} h_a \\ h_b + F(x)h_a \end{bmatrix} + \begin{bmatrix} D_a(x) \\ 0 \end{bmatrix} u$$

Differentiating the last rows with respect to time gives the expression

$$\underbrace{\begin{bmatrix} y_a \\ \phi(y, \dot{y}, x) \end{bmatrix}}_{Y^{(1)}} = \underbrace{\begin{bmatrix} h_a(x) \\ \tilde{h}_b(x) \end{bmatrix}}_{h^{(1)}} + \underbrace{\begin{bmatrix} D_a(x) \\ \tilde{D}_b(x) \end{bmatrix}}_{D^{(1)}} u$$

where

$$\begin{aligned} \phi(y, \dot{y}, x) &= \dot{y_b} + F\dot{y_a} + \sum y_{a,j} L_f F_j \\ \tilde{h}_b &= L_f(h_b + F h_a) \\ \tilde{D}_{b,j} &= L_{g_j}(h_b + F\tilde{h}_a - \sum y_{a,j} F_j) \end{aligned}$$

Here $y_{a,j}$ is the $j : th$ component of y_a, while g_j, F_j and $\tilde{D}_{b,j}$ denote the $j : th$ column of each matrix.

The process is repeated untill after k iterations there is an expression

$$Y^{(k)}(y, \dot{y}, \ldots, y^{(k)}, x) = h^{(k)}(x) + D^{(k)}(x)u$$

where $D^{(k)}(x)$ is nonsingular. The inverse is then

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}) + g(\hat{x})(D^{(k)}(\hat{x}))^{-1}(-h^{(k)}(\hat{x}) + Y^{(k)}) \\ u &= (D^{(k)}(\hat{x}))^{-1}(-h^{(k)}(\hat{x}) + Y^{(k)}) \end{aligned} \tag{11.24}$$

For linear systems this inversion procedure is called Silverman's algorithm (for linear systems F is a constant matrix which is a great simplification). The extension to nonlinear systems is called the Hirschorn-Singh algorithm.

Example 11.2 Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + u_1 \\ \dot{x}_2 &= u_2 \\ y &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} u \end{aligned} \tag{11.25}$$

Using

$$E_1 = I, \quad E_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

gives

$$\begin{bmatrix} y_1 \\ y_2 - y_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u$$

Differentiating the second row then produces

$$\begin{bmatrix} y_1 \\ \dot{y}_2 - \dot{y}_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1^3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} u$$

The D-matrix is now nonsingular so it is possible to solve for u and define the inverse.

$$
\begin{aligned}
u_1 &= -\hat{x}_1 + y_1 \\
u_2 &= \dot{y}_2 - \dot{y}_1 - \hat{x}_1^3 - \hat{x}_1 + y_1 \\
\dot{\hat{x}}_1 &= -\hat{x}_1^3 - \hat{x}_1 + y_1 \\
\dot{\hat{x}}_2 &= -\hat{x}_1^3 - \hat{x}_1 + y_1 + \dot{y}_2 - \dot{y}_1
\end{aligned}
$$

Note that the differential equation for $\hat{x}_2$ is redundant since we know that $\hat{x}_2 - \hat{x}_1 = y_2 - y_1$. ∎

Example 11.3 "The Hirschorn-Singh example".

$$
\begin{aligned}
\dot{x}_1 &= x_1 u_1 \\
\dot{x}_2 &= x_3 - x_3 u_1 \\
\dot{x}_3 &= x_1 u_2 \\
y_1 &= x_1 \\
y_2 &= x_2
\end{aligned}
$$

Since $D = 0$ the first step is to differentiate the whole y-vector.

$$
\dot{y} = \begin{bmatrix} 0 \\ x_3 \end{bmatrix} + \begin{bmatrix} x_1 & 0 \\ -x_3 & 0 \end{bmatrix} u
$$

Now

$$
E_1 = I, \quad E_2 = \begin{bmatrix} 1 & 0 \\ x_3/x_1 & 1 \end{bmatrix}
$$

so that multiplying with E_2 from the left gives

$$
\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 + \frac{x_3}{x_1}\dot{y}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ x_3 \end{bmatrix} + \begin{bmatrix} x_1 & 0 \\ 0 & 0 \end{bmatrix} u \tag{11.26}
$$

Differentiating the second output equation gives

$$
\ddot{y}_2 + \frac{x_3}{x_1}\ddot{y}_1 - \frac{x_3}{x_1}\dot{y}_1 u_1 + \dot{y}_1 u_2 = x_1 u_2 \tag{11.27}
$$

From the second row of (11.26) x_3 can be solved

$$
x_3 = -\frac{y_1 \dot{y}_2}{\dot{y}_1 - y_1}
$$

Since from definition $x_1 = y_1$, $x_2 = y_2$, all states are determined by the output and its derivatives, so the inverse will have no dynamics. Combining (11.26) and (11.27), u can be solved from

$$
\begin{aligned}
y_1 u_1 &= \dot{y}_1 \\
\frac{\dot{y}_1 \dot{y}_2}{\dot{y}_1 - y_1} u_1 + (\dot{y}_1 - y_1) u_2 &= \frac{\ddot{y}_1 \dot{y}_2}{\dot{y}_1 - y_1} - \ddot{y}_2
\end{aligned}
$$

Note that the inverse can only be computed for $y_1 \neq 0$ and $\dot{y}_1 \neq y_1$.

∎