When is a graph a state space description?

S. T. Glad

I. Introduction

Why graphs?

Many physical systems can be described as interconnections of simple subsystems.

The interconnections form a discrete structure.

This structure can often be described by a graph.

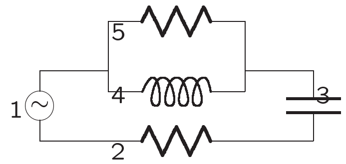
Examples:

1. electric circuits (Kirchhoff 1847)

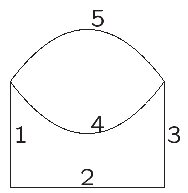
2. linear graphs for lumped parameter systems (Trent 1955)

3. bond graphs (Paynter ca 1955)

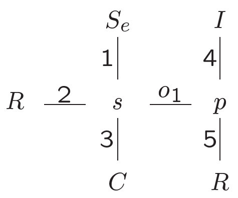
Example 1: a simple electric circuit



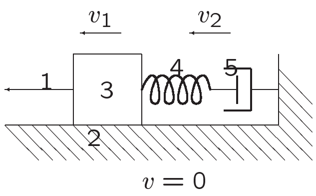
Corresponding linear graph:



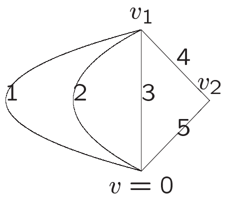
Corresponding bond graph:



Example 2: a mechanical lumped parameter system

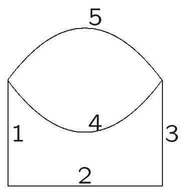


Corresponding linear graph:

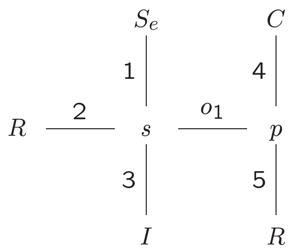


The nodes correspond to velocities. The flows along edges correspond to forces.

One could also consider the dual graph (vertices and edges, flows and efforts change places)



Converting this linear graph of the mechanical system to a bond graph using "the electric method" gives



Much (all?) of the modeling that is done with bond graphs could be done with linear graphs.

The mathematical theory of linear graphs is well developed and understood (mostly through work in circuit theory).

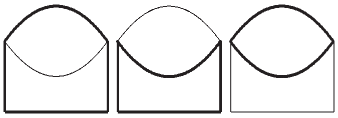
Bond graphs have been developed heuristically and a firm mathematical understanding is still lacking for parts of the theory.

II. Linear graphs

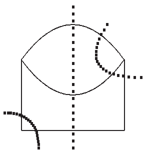
An extension of circuit theory

Fundamental concepts for linear graphs:

Circuits:



Cut sets:



(not all possibilities are shown)

Relationships between efforts and flows in linear graphs.

"Kirchhoff's voltage law" The sum of all efforts along a circuit is zero.

"Kirchhoff's current law" The sum of all flows over a cut set is zero.

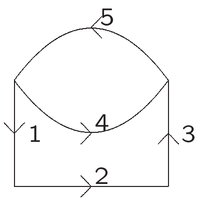
The formulation of these rules requires a reference direction for each edge of the graph.

Matrix formulation.

$$Ce = 0, \quad Qf = 0$$

where e is a vector of all efforts, f is a vector of all flows, C is the **circuit matrix** and Q the **cut set matrix**.

Example:



$$\begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{pmatrix} = 0$$

Each row corresponds to a circuit. Note the linear dependence.

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} = 0$$

Each row corresponds to a cut set.

To get linearly independent equations one can use fundamental circuits and fundamental cut sets.

Choose a spanning tree T (a set of edges, containing no circuit, connecting all nodes).

Each edge, not in T, defines (together with some edges in T) a circuit. They form together the fundamental circuits corresponding to T.

Each edge in T defines (together with some edges outside T) a cut set. They form together the fundamental cut sets corresponding to T.

The circuit and cut set matrices have the form (after rearranging the columns)

$$C = \begin{pmatrix} C_1 & I_n \end{pmatrix}, \quad Q = \begin{pmatrix} I_m & Q_1 \end{pmatrix}$$

Consider the equations

$$\begin{pmatrix} C_1 & I_n \end{pmatrix} \begin{pmatrix} e_m \\ e_n \end{pmatrix} = 0, \quad \begin{pmatrix} I_m & Q_1 \end{pmatrix} \begin{pmatrix} f_m \\ f_n \end{pmatrix} = 0$$

and the total power flow, which has to be zero

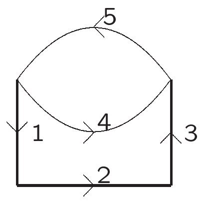
$$P = e_m^T f_m + e_n^T f_n = 0$$

It follows that one must have

$$C_1 = -Q_1^T$$

(which can also be shown graph-theoretically)

Example.



Solid edges mark the spanning tree. The circuit and cut set matrices are

$$C = \begin{pmatrix} -1 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}$$

III. When and how can you get a state space description out of a linear graph?

Consider a physical system where the interconnection structure for flows and efforts is given by a linear graph. Let the elementary subsystems be

- R-elements: $e = Rf$
- I-elements: $L\dot{f} = e$
- C-elements: $C\dot{e} = f$
- effort sources: e given
- flow sources: f given

- x:
  all efforts of C-elements
  all flows of I-elements

- u:
  all efforts of effort sources
  all flows of flow sources.

- y:
  all other variables

When do we have

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

"a canonical state space description"?

Necessary conditions for a canonical state space description:

- There is no circuit consisting exclusively of effort sources and C-elements.

- There is no cut set consisting exclusively of flow sources and I-elements.

Equivalent condition:

There exists a spanning tree containing all effort sources and C-elements and no flow sources or I-elements.

The necessary condition also turns out to be sufficient:

Theorem 1
A linear graph consisting of effort- and flow sources, linear I- and C-elements and linear positive R-elements has a canonical state space description if and only if there exists a spanning tree containing all effort sources and C-elements and no flow sources or I-elements.

Static case proved by Kirchhoff 1847; extends trivially to general case.

If R-elements of arbitrary sign are present, the result still holds for "almost all" resistance values.

The proof actually constructs the A-, B-, C- and D-matrices explicitly.

Let the spanning tree be denoted T and let

$$e = \begin{pmatrix} e_E & e_C & e_T & e_R & e_I & e_F \end{pmatrix}^T$$
$$f = \begin{pmatrix} f_E & f_C & f_T & f_R & f_I & f_F \end{pmatrix}^T$$

be the efforts and flows associated with the effort sources, C-elements, R-elements in T, R-elements outside T, I-elements and flow sources respectively.

The circuit and cut set equations are then given by

$$\begin{pmatrix} C_{RE} & C_{RC} & C_{RT} & I & 0 & 0 \\ C_{IE} & C_{IC} & C_{IT} & 0 & I & 0 \\ C_{FE} & C_{FC} & C_{FT} & 0 & 0 & I \end{pmatrix} e = 0$$
$$\begin{pmatrix} I & 0 & 0 & Q_{ER} & Q_{EI} & Q_{EF} \\ 0 & I & 0 & Q_{CR} & Q_{CI} & Q_{CF} \\ 0 & 0 & I & Q_{TR} & Q_{TI} & Q_{TF} \end{pmatrix} f = 0$$

and the element equations are

$$D_I \dot{f}_I = e_I, \quad D_C \dot{e}_C = f_C$$
$$e_T = D_T f_T, \quad e_R = D_R f_R$$

where D_I, D_C, D_T and D_R are diagonal matrices containing the component values.

The equations give

$$\begin{aligned} D_R f_R = e_R = & -C_{RE}e_E - C_{RC}e_C - C_{RT}e_T = \\ & - C_{RE}e_E - C_{RC}e_C + \\ & C_{RT}D_T(Q_{TR}f_R + Q_{TI}f_I + Q_{TF}f_F) \end{aligned}$$

Solving for f_R gives

$$(D_R - C_{RT}D_TQ_{TR})f_R = \cdots$$

Since $C_{RT} = -Q_{TR}^T$

$$(D_R + C_{RT}D_TC_{RT}^T)f_R = \cdots$$

D_R, D_T diagonal with positive elements $\Rightarrow$

$$M = D_R + C_{RT}D_TC_{RT}^T > 0$$

i.e. M is non-singular.

Thus all efforts and flows can be expressed in e_E, e_C, f_I and f_F, in particular the right hand sides of $\dot{e}_C$ and $\dot{f}_I$.

The state space description is of the form

$$\frac{d}{dt}\begin{pmatrix} e_C \\ f_I \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} e_C \\ f_I \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} e_E \\ f_F \end{pmatrix}$$

$$y = \begin{pmatrix} C_{11} & C_{12} \end{pmatrix}\begin{pmatrix} e_C \\ f_I \end{pmatrix} + \begin{pmatrix} D_{11} & D_{12} \end{pmatrix}\begin{pmatrix} e_E \\ f_F \end{pmatrix}$$

where e.g.

$$A_{11} = D_C^{-1}Q_{CR}M^{-1}C_{RC}$$

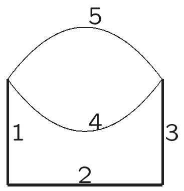
The theorem can be generalized:

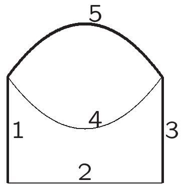
Theorem 2
Consider a linear graph consisting of effort- and flow sources, linear I- and C-elements and linear positive R-elements. If T is a spanning tree containing all effort sources and no flow sources and with the quantity

(nr of C:s in T) + (nr of I:s outside T)

maximal, then the efforts of C-elements in T and flows of I-elements outside T can be chosen as state variables. The effort and flow sources define the inputs, but possibly there will be derivatives of the input variables in the equations.

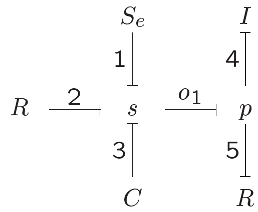
Consider again the electrical circuit example. There are two choices of spanning trees that show the existence of a canonical state space description:

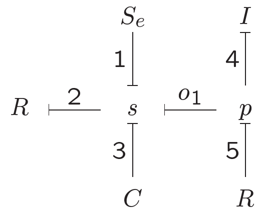




IV. Relations between linear graphs and bond graphs

The two choices of spanning trees for the circuit example correspond to two different choices of causality in the corresponding bond graph:

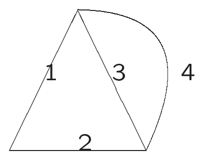




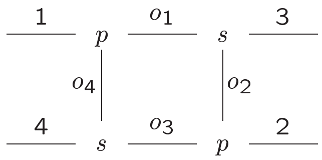
For large classes of systems there is a one-to-one correspondence between (equivalence classes of) linear graphs and (equivalence classes of) bond graphs.

The mathematical "reason" seems to be that typical interconnection structures are so called matroids. Linear graphs or bond graphs are just different "graphical illustrations" of matroids. (Birkett, ca 1990)

Not all aspects of the relations between linear graphs and bond graphs are well understood. The linear graph



can be considered equivalent to the bond graph



only for certain choices of the power directions in the internal bonds $o_1 - o_4$. An even number of them have to be clockwise.

What is the interpretation of those cases that do not correspond to a linear graph?

V. Algorithmic aspects

How do you construct the required spanning trees?

There is a “greedy algorithm” for constructing the minimal spanning tree of a graph.

VI. Extensions and Conclusions

Much of the theory can be extended to systems with transformers, gyrators, multi-port elements and nonlinear, monotone resistors.

There is still some research to be done on the fundamental aspects of linear graphs and bond graphs.